Solution of Equations by Iteration

We begin with methods of finding solutions of a single equation

(1) $$f(x) = 0$$

where f is a given function.

A **solution** of (1) is a value $x = s$ such that $f(s) = 0$.

Fixed-Point Iteration

In one way or another we transform (1) *algebraically* into the form

(2) $$x = g(x).$$

Then we choose an x_0 and compute $x_1 = g(x_0)$, $x_2 = g(x_1)$, and in general

(3) $$x_{n+1} = g(x_n) \qquad (n = 0, 1, \cdots).$$

continued

- A solution of (2) is called a **fixed point** of g, motivating the name of the method. This is a solution of (1), since from $x = g(x)$ we can return to the original form $f(x) = 0$.
- From (1) we may get several different forms of (2). The behavior of corresponding iterative sequences $x_0, x_1, \cdots$ may differ, in particular, with respect to their speed of convergence.
- Indeed, some of them may not converge at all.

EXAMPLE1

$f(x) = x^2 - 3x + 1 = 0$. We know the solutions are

$$x = 1.5 \pm \sqrt{1.25}, \qquad \text{thus} \qquad 2.618\,034 \qquad \text{and} \qquad 0.381\,966,$$

Solution. The equation may be written

(4a)

$$x = g_1(x) = \tfrac{1}{3}(x^2 + 1), \qquad \text{thus} \qquad x_{n+1} = \tfrac{1}{3}(x_n{}^2 + 1).$$

continued

Function (4a)

(1) If we choose $x_0 = 1$, we obtain the sequence

$$x_0 = 1.000, \quad x_1 = 0.667, \quad x_2 = 0.481, \quad x_3 = 0.411, \quad x_4 = 0.390, \cdots$$

which seems to approach the smaller solution.

(2) If we choose $x_0 = 2$, the situation is similar.

(3) If we choose $x_0 = 3$, we obtain the sequence

$$x_0 = 3.000, \quad x_1 = 3.333, \quad x_2 = 4.037, \quad x_3 = 5.766, \quad x_4 = 11.415, \cdots$$

which diverges.

continued

Function (4b)

Our equation may also be written (divide by *x*)

(4b) $$x = g_2(x) = 3 - \frac{1}{x}, \qquad \text{thus} \qquad x_{n+1} = 3 - \frac{1}{x_n},$$

(1) If we choose $x_0 = 1$, we obtain the sequence

$$x_0 = 1.000, \quad x_1 = 2.000, \quad x_2 = 2.500, \quad x_3 = 2.600, \quad x_4 = 2.615, \cdots$$

which seems to approach the larger solution.

(2) If we choose $x_0 = 3$, we obtain the sequence

$$x_0 = 3.000, \quad x_1 = 2.667, \quad x_2 = 2.625, \quad x_3 = 2.619, \quad x_4 = 2.618, \cdots.$$

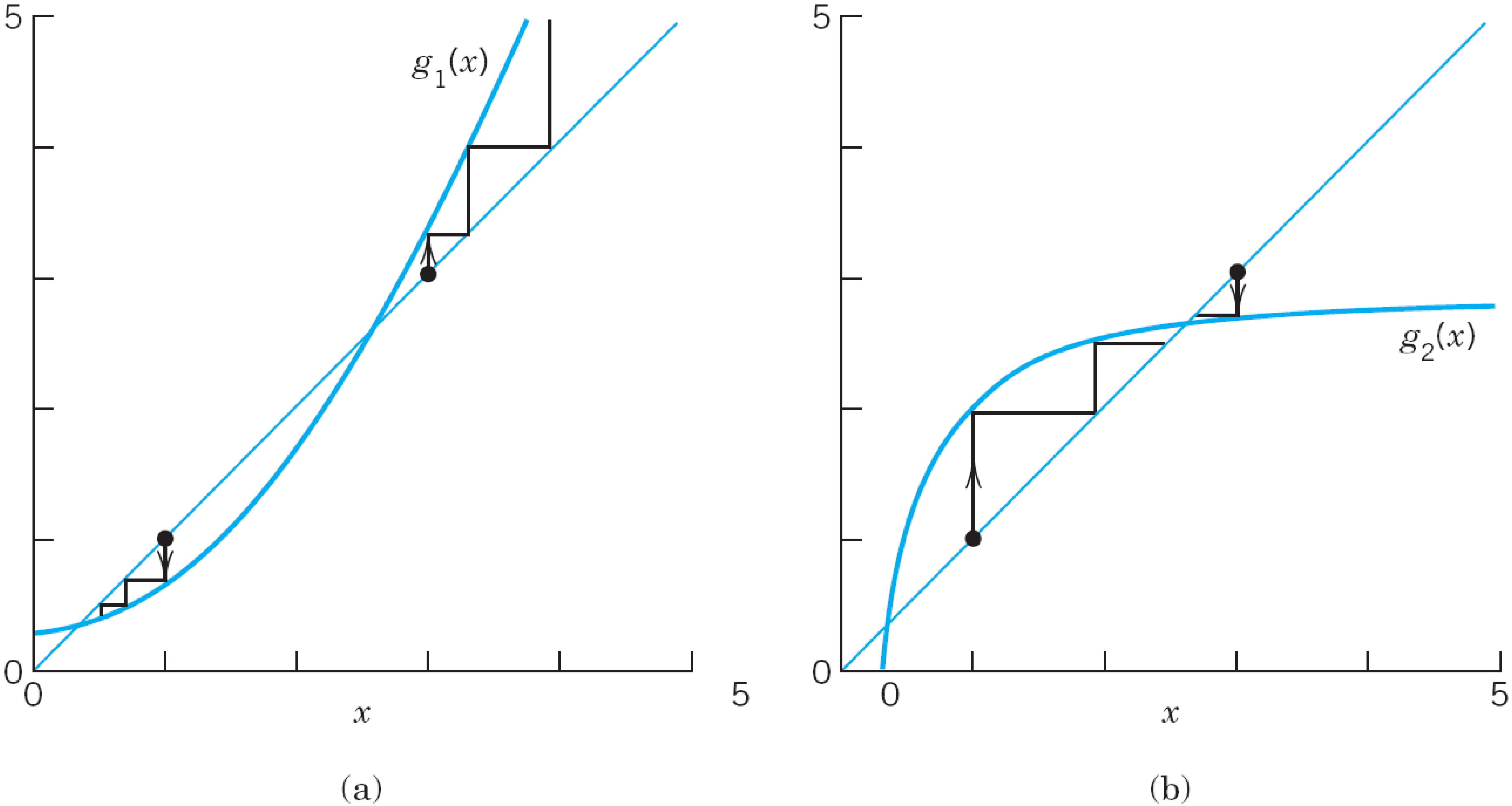
continued

Observations

- Our figures show the following. In the lower part of Fig. 423a the slope of $g_1(x)$ is less than the slope of $y = x$, which is 1, thus $|g'_1(x)| < 1$, and we seem to have convergence. In the upper part, $g_1(x)$ is steeper ($g'_1(x) > 1$) and we have divergence.
- In Fig. 423b the slope of $g_2(x)$ is less near the intersection point ($x = 2.618$, fixed point of g_2, solution of $f(x) = 0$), and both sequences seem to converge.
- From all this we conclude that convergence seems to depend on the fact that in a neighborhood of a solution the curve of $g(x)$ is less steep than the straight line $y = x$, and we shall now see that this condition $|g'(x)| < 1$ (= slope of $y = x$) is sufficient for convergence.

continued

Fig. 423. Example 1, iterations (4a) and (4b)



THEOREM 1

Convergence of Fixed-Point Iteration

Let $x = s$ be a solution of $x = g(x)$ and suppose that g has a continuous derivative in some interval J containing s.

Then if $|g'(x)| \leq K < 1$ in J, the iteration process defined by (3) converges for any x_0 in J, and the limit of the sequence $\{x_n\}$ is s.

EXAMPLE2

$$f(x) = x^3 + x = 1 = 0$$

Solution. A sketch shows that a solution lies near $x = 1$. We may write the equation as $(x^2 + 1)x = 1$ or

$$x = g_1(x) = \frac{1}{1 + x^2}, \qquad \text{so that} \qquad x_{n+1} = \frac{1}{1 + {x_n}^2}. \qquad \text{Also} \qquad |g_1'(x)| = \frac{2|x|}{(1 + x^2)^2} < 1$$

for any x because $4x^2/(1 + x^2)^4 = 4x^2/(1 + 4x^2 + \cdots\cdot) < 1$, so that by Theorem 1 we have convergence for any x_0. Choosing $x_0 = 1$, we obtain (Fig. 424)

$$x_1 = 0.500, \quad x_2 = 0.800, \quad x_3 = 0.610, \quad x_4 = 0.729, \quad x_5 = 0.653, \quad x_6 = 0.701, \cdots .$$

The solution is $s = 0.682\,328$.

continued

The given equation may also be written

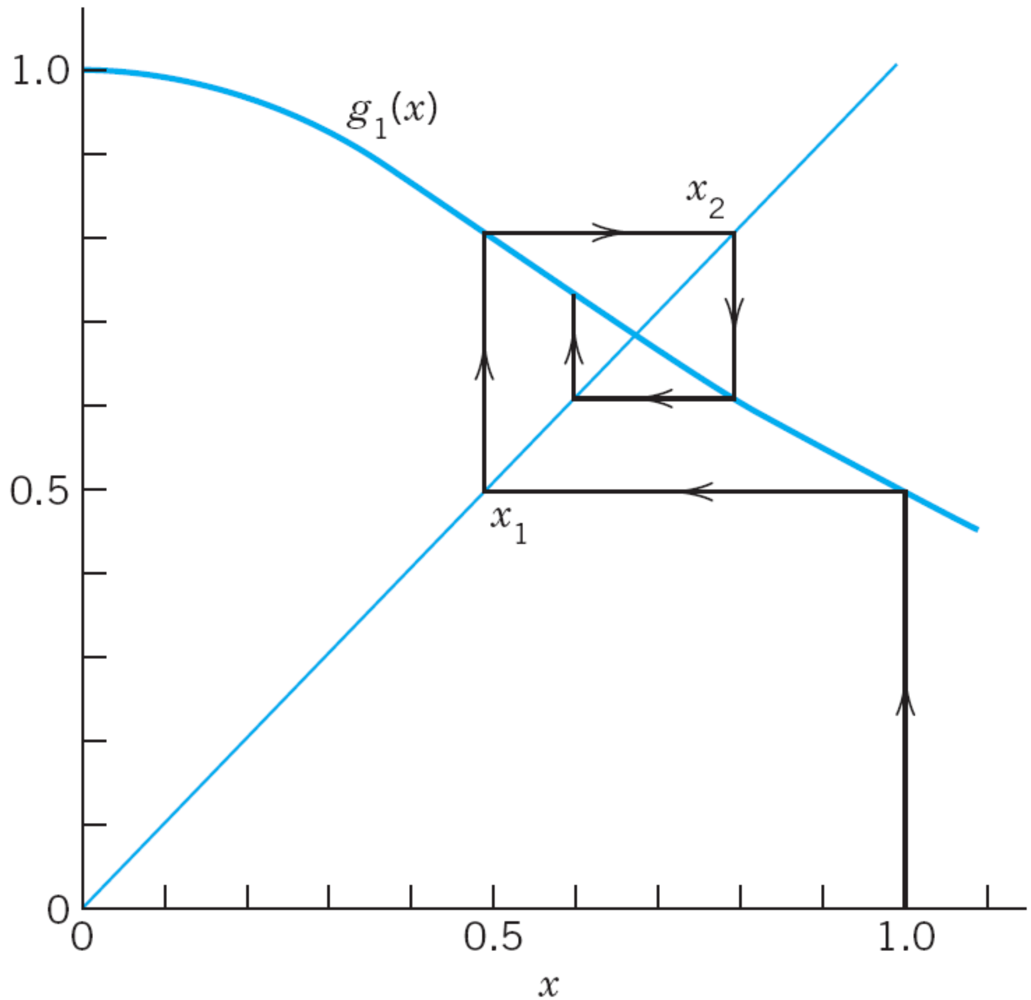
$$x = g_2(x) = 1 - x^3. \qquad \text{Then} \qquad |g_2'(x)| = 3x^2$$

and this is greater than 1 near the solution, so that we cannot apply Theorem 1 and assert convergence. Try $x_0 = 1$, $x_0 = 0.5$, $x_0 = 2$ and see what happens.

The example shows that the transformation of a given $f(x) = 0$ into the form $x = g(x)$ with g satisfying $|g'(x)| \le K < 1$ may need some experimentation.

continued

Fig. 424. Iteration in Example 2



Newton's Method

Newton's method, also known as **Newton–Raphson's method,** is another iteration method for solving equations $f(x) = 0$, where f is assumed to have a continuous derivative f'.

The underlying idea is that we approximate the graph of f by suitable tangents. Using an approximate value x_0 obtained from the graph of f, we let x_1 be the point of intersection of the x-axis and the tangent to the curve of f at x_0 (see Fig. 425). Then

$$\tan \beta = f'(x_0) = \frac{f(x_0)}{x_0 - x_1}, \qquad \text{hence} \qquad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

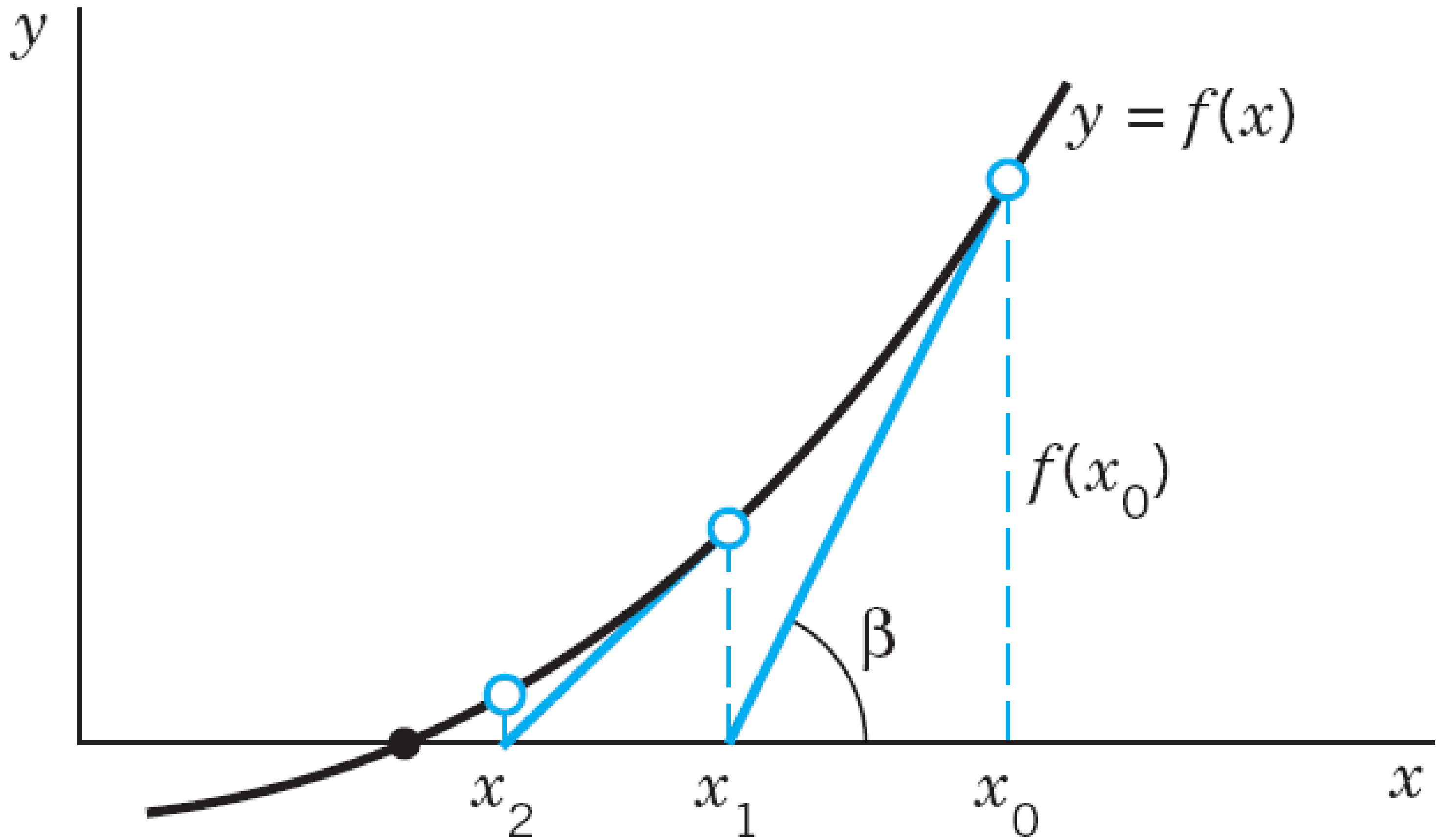
General Formula

One can algebraically solve the approximated Taylor's expansion

$$(5) \qquad f(x_{n+1}) \approx f(x_n) + (x_{n+1} - x_n)f'(x_n) = 0.$$

continued

Fig. 425. Newton's method



ALGORITHM NEWTON (f, f', x_0, ϵ, N)

This algorithm computes a solution of $f(x) = 0$ given an initial approximation x_0 (starting value of the iteration). Here the function $f(x)$ is continuous and has a continuous derivative $f'(x)$.

INPUT: f, f', initial approximation x_0, tolerance $\epsilon > 0$, maximum number of iterations N.

OUTPUT: Approximate solution x_n ($n \leqq N$) or message of failure.

continued

Pseudo Code

For $n = 0, 1, 2, \cdots, N-1$ do:

1 Compute $f'(x_n)$.

2 If $f'(x_n) = 0$ then OUTPUT "Failure". Stop.

[*Procedure completed unsuccessfully*]

3 Else compute

(5)
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} .$$

4 If $|x_{n+1} - x_n| \leqq \epsilon |x_n|$ then OUTPUT x_{n+1}. Stop.

[*Procedure completed successfully*]

End

5 OUTPUT "Failure". Stop.

[*Procedure completed unsuccessfully after N iterations*]

End NEWTON

EXAMPLE3

Compute the square root x of a given positive number c and apply it to $c = 2$.

Solution. We have $x = \sqrt{c}$, hence $f(x) = x^2 - c = 0$, $f'(x) = 2x$, and (5) takes the form

$$x_{n+1} = x_n - \frac{x_n{}^2 - c}{2x_n} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right).$$

For $c = 2$, choosing $x_0 = 1$, we obtain

$$x_1 = 1.500\,000, \qquad x_2 = 1.416\,667, \qquad x_3 = 1.414\,216, \qquad x_4 = 1.414\,214, \cdots.$$

x_4 is exact to 6D.

EXAMPLE

$$2 \sin x = x$$

Solution.

Setting $f(x) = x - 2 \sin x$, we have $f'(x) = 1 - 2 \cos x$, and (5) gives

$$x_{n+1} = x_n - \frac{x_n - 2 \sin x_n}{1 - 2 \cos x_n} = \frac{2(\sin x_n - x_n \cos x_n)}{1 - 2 \cos x_n} = \frac{N_n}{D_n}.$$

continued

From the graph of f we conclude that the solution is near $x_0 = 2$. We compute:

n	x_n	N_n	D_n	x_{n+1}
0	2.00000	3.48318	1.83229	1.90100
1	1.90100	3.12470	1.64847	1.89552
2	1.89552	3.10500	1.63809	1.89550
3	1.89550	3.10493	1.63806	1.89549

$x_4 = 1.89549$ is exact to 5D

the exact solution to 6D is 1.895 494.

EXAMPLE 5

$$f(x) = x^3 + x - 1 = 0$$

Solution. From (5) we have

$$x_{n+1} = x_n - \frac{x_n^{\,3} + x_n - 1}{3x_n^{\,2} + 1} = \frac{2x_n^{\,3} + 1}{3x_n^{\,2} + 1}.$$

Starting from $x_0 = 1$, we obtain

$$x_1 = 0.750\,000, \qquad x_2 = 0.686\,047, \qquad x_3 = 0.682\,340, \qquad x_4 = 0.682\,328, \cdots$$

where x_4 has the error $-1 \cdot 10^{-6}$. A comparison with Example 2 shows that the present convergence is much more rapid.

Difficulties in Newton's Method

Difficulties may arise if $|f'(x)|$ is very small near a solution s of $f(x) = 0$. Geometrically, small $|f'(x)|$ means that the tangent of $f(x)$ near s almost coincides with the x-axis (so that double precision may be needed to get $f(x)$ and $f'(x)$ accurately enough).

In this case we call the equation $f(x) = 0$ **ill-conditioned.**

continued

EXAMPLE 6 An Ill-Conditioned Equation

- $f(x) = x^5 + 10^{-4}x = 0$ is ill-conditioned. $x = 0$ is a solution. $f'(0) = 10^{-4}$ is small. At $\tilde{s} = 0.1$ the residual $f(0.1) = 2 \cdot 10^{-5}$ is small, but the error –0.1 is larger in absolute value by a factor 5000. Invent a more drastic example of your own.

Secant Method

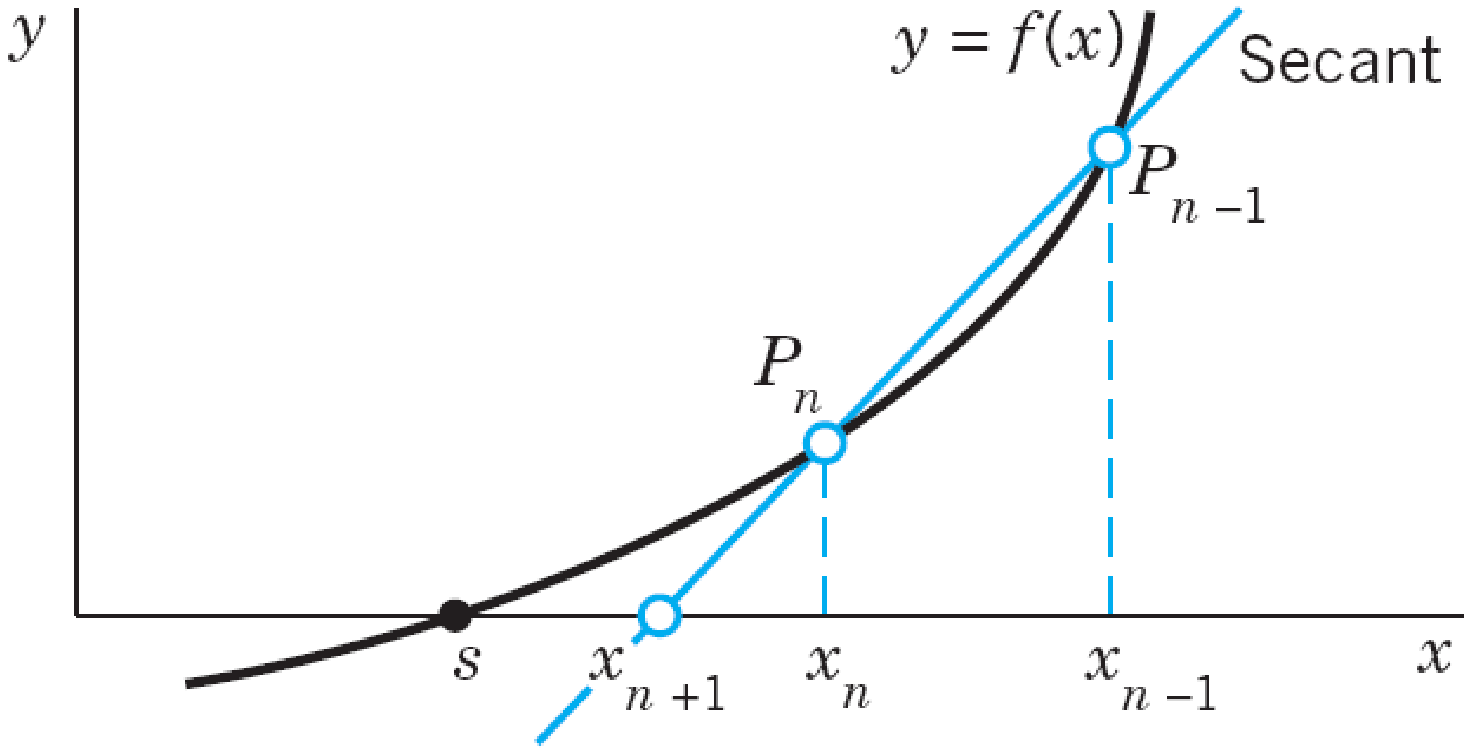
Newton's method is very powerful but has the disadvantage that the derivative f' may sometimes be a far more difficult expression than f itself and its evaluation therefore computationally expensive.

This situation suggests the idea of replacing the derivative with the difference quotient

$$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} .$$

continued

Fig. 426. Secant method



Then instead of (5) we have the formula of the popular secant method

(10)

$$x_{n+1} = x_n - f(x_n)\,\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\,.$$

continued

Geometrically, we intersect the x-axis at x_{n+1} with the secant of $f(x)$ passing through $P_{n\text{-}1}$ and P_n in Fig. 426.

We need two starting values x_0 and x_1.

Evaluation of derivatives is now avoided.

It can be shown that convergence is almost like Newton's method. The algorithm is similar to that of Newton's method.

EXAMPLE 8 Secant Method

Solve $f(x) = x - 2 \sin x = 0$ by the secant method, starting from $x_0 = 2$, $x_1 = 1.9$.

Solution. Here, (10) is

$$x_{n+1} = x_n - \frac{(x_n - 2 \sin x_n)(x_n - x_{n-1})}{x_n - x_{n-1} + 2(\sin x_{n-1} - \sin x_n)} = x_n - \frac{N_n}{D_n} .$$

continued

Numerical values are:

n	x_{n-1}	x_n	N_n	D_n	$x_{n+1} - x_n$
1	2.000 000	1.900 000	−0.000 740	−0.174 005	−0.004 253
2	1.900 000	1.895 747	−0.000 002	−0.006 986	−0.000 252
3	1.895 747	1.895 494	0		0

x_3 = 1.895 494 is exact to 6D. See Example 4.