LAPLACE TRANSFORMS

1 Introduction

Let f(t) be a given function which is defined for all positive values of t, if

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

exists, then F(s) is called *Laplace transform* of f(t) and is denoted by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

The inverse transform, or inverse of $\mathcal{L}\{f(t)\}$ or F(s), is

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

where *s is a positive real number or a complex number with positive real part*.

[Examples]

$$\mathcal{L}\{1\} = \frac{1}{s} \quad ; \quad \mathcal{L}\{ e^{at} \} = \frac{1}{s - a}$$

$$\mathcal{L}\{ \cos \omega t \} = \int_0^\infty e^{-st} \cos \omega t \, dt = \left. \frac{e^{-st} (-s \cos \omega t + \omega \sin \omega t)}{\omega^2 + s^2} \right|_{t=0}^{\infty} = \frac{s}{s^2 + \omega^2}$$

(Note that $s > 0$, otherwise $e^{-st} \Big|_{t=\infty}$ diverges)

$$\mathcal{L}\{ \sin \omega t \} = \int_0^\infty e^{-st} \sin \omega t \, dt \qquad \text{(integration by parts)}$$

$$= \left. \frac{- e^{-st} \sin \omega t}{s} \right|_{t=0}^{\infty} + \frac{\omega}{s} \int_0^\infty e^{-st} \cos \omega t \, dt = \frac{\omega}{s} \int_0^\infty e^{-st} \cos \omega t \, dt = \frac{\omega}{s} \mathcal{L}\{ \cos \omega t \} = \frac{\omega}{s^2 + \omega^2}$$

$$\mathcal{L}\{ t^n \} = \int_0^\infty t^n e^{-st} dt \qquad (\text{let } t = z/s, \ dt = dz/s)$$

$$= \int_0^\infty \left[\frac{z}{s} \right]^n e^{-z} \frac{dz}{s} = \frac{1}{s^{n+1}} \int_0^\infty z^n e^{-z} dz = \frac{\Gamma(n+1)}{s^{n+1}} \qquad \left(\text{Recall } \Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt \right)$$

If $n = 1, 2, 3, \ldots \Gamma(n+1) = n!$ $\Rightarrow$ $\mathcal{L}\{ t^n \} = \frac{n!}{s^{n+1}}$ where n is a positive integer

[Theorem] Linearity of the Laplace Transform

$$\mathcal{L}\{ a\, f(t) + b\, g(t) \} = a\,\mathcal{L}\{ f(t) \} + b\,\mathcal{L}\{ g(t) \}$$

where a and b are constants.

[Example] If $\mathcal{L}\{ e^{at} \} = \dfrac{1}{s - a}$, then $\mathcal{L}\{ \sinh at \} = ??$

Since

$$\mathcal{L}\{ \sinh at \} = \mathcal{L}\left\{ \frac{e^{at} - e^{-at}}{2} \right\} = \frac{1}{2}\mathcal{L}\{ e^{at} \} - \frac{1}{2}\mathcal{L}\{ e^{-at} \} = \frac{1}{2}\left\{ \frac{1}{s-a} - \frac{1}{s+a} \right\} = \frac{a}{s^2 - a^2}$$

[Example] Find $\mathcal{L}^{-1}\left\{ \dfrac{s}{s^2 - a^2} \right\}$

$$\mathcal{L}^{-1}\left\{ \frac{s}{s^2 - a^2} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a} \right] \right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{ \frac{1}{s-a} \right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{ \frac{1}{s+a} \right\}$$

$$= \frac{1}{2} e^{at} + \frac{1}{2} e^{-at} = \frac{e^{at} + e^{-at}}{2} = \cosh at$$

Existence of Laplace Transforms

[Example] $\mathcal{L}\{1/t\}$

From definition,

$$\mathcal{L}\{1/t\} = \int_0^\infty \frac{e^{-st}}{t}\,dt = \int_0^1 \frac{e^{-st}}{t}\,dt + \int_1^\infty \frac{e^{-st}}{t}\,dt$$

But for t in the interval **$0 \le t \le 1$**, $e^{-st} \ge e^{-s}$ (if $s > 0$), then $\int_0^\infty \frac{e^{-st}}{t}\,dt \ge e^{-s}\int_0^1 \frac{dt}{t} + \int_1^\infty \frac{e^{-st}}{t}\,dt$

However, $\int_0^1 t^{-1}\,dt = \lim_{A\to 0}\int_A^1 t^{-1}\,dt = \lim_{A\to 0} \ln t \Big|_A^1 = \lim_{A\to 0}(\ln 1 - \ln A) = \lim_{A\to 0}(-\ln A) = \infty$

$\Rightarrow \int_0^\infty \frac{e^{-st}}{t}\,dt$ diverges, $\Rightarrow$ no Laplace Transform for $1/t$!

Piecewise Continuous Functions

A function is called **piecewise continuous** in an interval $a \le t \le b$ if the interval can be subdivided into a finite number of intervals in each of which (1) the function is continuous and (2) has finite right- and left-hand limits.

Existence Theorem

(Sufficient Conditions for Existence of Laplace Transforms)

Let f be ***piecewise continuous*** on $t \ge 0$ and satisfy the condition $|f(t)| \le M e^{\gamma t}$

for fixed non-negative constants γ and M, then $\mathcal{L}\{ f(t) \}$ exists for all $s > \gamma$.

[Proof]

Since f(t) is piecewise continuous, $e^{-st} f(t)$ is **integratable** over any finite interval on $t > 0$,

$$|\mathcal{L}\{ f(t) \}| = \left| \int_0^\infty e^{-st} f(t)\, dt \right| \le \int_0^\infty e^{-st} |f(t)|\, dt \le \int_0^\infty M e^{\gamma t} e^{-st}\, dt = \frac{M}{s-\gamma} \quad \textbf{if } \mathrm{Re}(s) > \gamma$$

$$\Rightarrow \quad \mathcal{L}\{ f(t) \} \text{ exists.}$$

[Examples] Do $\mathcal{L}\{ t^n \}, \mathcal{L}\{ e^{t^2} \}, \mathcal{L}\{ t^{-1/2} \}$ exist?

(i) $$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots + \frac{t^n}{n!} + \ldots$$

$$\Rightarrow \quad t^n \le n!\, e^t$$

$$\Rightarrow \quad \mathcal{L}\{ t^n \} \text{ exists.}$$

(ii) $e^{t^2} > M e^{\gamma t}$ as t approaches infinity

$\Rightarrow$ $\mathcal{L}\{ e^{t^2} \}$ may not exist.

(iii) $\mathcal{L}\{ t^{-1/2} \} = \sqrt{\frac{\pi}{s}}$ exists, but note that $t^{-1/2} \to \infty$ for $t \to 0$!

(See item 4 on page 249 in textbook!)

2 Some Important Properties of Laplace Transforms

(1) Linearity Properties

$$\mathcal{L}\{ a\, f(t) + b\, g(t) \} = a\,\mathcal{L}\{ f(t) \} + b\,\mathcal{L}\{ g(t) \}$$

where a and b are constants. (i.e., Laplace transform operator is linear)

(2) Laplace Transform of Derivatives

If f(t) is *continuous* and f'(t) is *piecewise continuous* for $t \geq 0$, then

$$\mathcal{L}\{ f'(t) \} = s\,\mathcal{L}\{ f(t) \} - f(0^+)$$

[Proof]

$$\mathcal{L}\{ f'(t) \} = \int_0^\infty f'(t)\, e^{-st}\, dt$$

Integration by parts by letting

$u = e^{-st}$ $dv = f'(t)\, dt$

$du = -s\, e^{-st}\, dt$ $v = f(t)$

$$\Rightarrow \quad \mathcal{L}\{ f'(t) \} = \left[e^{-st} f(t) \right]_0^\infty + s\int_0^\infty e^{-st} f(t) dt = -f(0) + s\,\mathrm{L}\{ f(t) \} \quad \Rightarrow \quad \boxed{\mathcal{L}\{ f'(t) \} = s\,\mathcal{L}\{ f(t) \} - f(0^+)}$$

Theorem: $f(t)$, $f'(t)$, . . ., $f^{(n-1)}(t)$ are continuous functions for $t \geq 0$, and $f^{(n)}(t)$ is piecewise continuous function, then

$$\mathcal{L}\{ f^{(n)} \} = s^n \mathcal{L}\{ f \} - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0)$$

e.g, $\mathcal{L}\{ f''(t) \} = s^2 \mathcal{L}\{ f(t) \} - s f(0) - f'(0)$ and $\mathcal{L}\{ f'''(t) \} = s^3 \mathcal{L}\{ f(t) \} - s^2 f(0) - s f'(0) - f''(0)$

[Example] $\mathcal{L}\{ e^{at} \} = ??$

$f(t) = e^{at}$, $f(0) = 1$ and $f'(t) = a\, e^{at}$

$$\Rightarrow \quad \mathcal{L}\{ f'(t) \} = s \mathcal{L}\{ f(t) \} - f(0) \quad \text{or} \quad \mathcal{L}\{ a\, e^{at} \} = s \mathcal{L}\{ e^{at} \} - 1 \quad \text{or} \quad a \mathcal{L}\{ e^{at} \} = s \mathcal{L}\{ e^{at} \} - 1$$

$$\Rightarrow \quad \mathcal{L}\{ e^{at} \} = \frac{1}{s - a}$$

[Example] $\mathcal{L}\{ \sin at \} = ??$

$f(t) = \sin at$, $f(0) = 0$

$f'(t) = a \cos at$, $f'(0) = a$

$f''(t) = -a^2 \sin at$

$$\mathcal{L}\{ f''(t) \} = s^2 \mathcal{L}\{ f(t) \} - s f(0) - f'(0) \quad \Rightarrow \quad \mathcal{L}\{ -a^2 \sin at \} = s^2 \mathcal{L}\{ \sin at \} - s \times 0 - a$$

$$\text{or} \quad -a^2 \mathcal{L}\{ \sin at \} = s^2 \mathcal{L}\{ \sin at \} - a \quad \Rightarrow \quad \mathcal{L}\{ \sin at \} = \frac{a}{s^2 + a^2}$$

[Example] $\mathcal{L}\{\sin^2 t\} = \dfrac{2}{s(s^2+4)}$

Known: $f(t) = \sin^2 t$; $f(0) = 0$; $f'(t) = 2\sin t \cos t = \sin 2t$

Also, $\mathrm{L}\{\sin 2t\} = \dfrac{2}{s^2+4}$

Thus, $\mathrm{L}\{\sin 2t\} = \mathrm{L}\{f'\} = s\,\mathrm{L}\{f\} - f(0) = s\,\mathrm{L}\{\sin^2 t\}$

$$\mathrm{L}\{\sin^2 t\} = \frac{1}{s}\mathrm{L}\{\sin 2t\} = \frac{2}{s(s^2+4)}$$

[Example] $\mathcal{L}\{f(t)\} = \mathcal{L}\{t \sin \omega t\} = \dfrac{2\omega s}{(s^2+\omega^2)^2}$

$f(t) = t\sin \omega t, \qquad f(0) = 0$

$f'(t) = \sin \omega t + \omega t \cos \omega t, \qquad f'(0) = 0$

$f''(t) = 2\omega \cos \omega t - \omega^2 t \sin \omega t = 2\omega \cos \omega t - \omega^2 f(t)$

$\mathrm{L}\{f''\} = 2\omega\,\mathrm{L}\{\cos \omega t\} - \omega^2\,\mathrm{L}\{f(t)\} = s^2\,\mathrm{L}\{f\} - sf(0) - f'(0) = s^2\,\mathrm{L}\{f\}$

$$(s^2+\omega^2)\,\mathrm{L}\{f\} = 2\omega \frac{s}{s^2+\omega^2} \Rightarrow \mathrm{L}\{t \sin \omega t\} = \frac{2\omega s}{(s^2+\omega^2)^2}$$

[Example] $y'' - 4y = 0$, $y(0) = 1$, $y'(0) = 2$ **(IVP!)**

[Solution] Take Laplace Transform on both sides,

$\mathcal{L}\{y'' - 4y\} = \mathcal{L}\{0\}$ or $\mathcal{L}\{y''\} - 4\mathcal{L}\{y\} = 0$

$s^2\mathcal{L}\{y\} - s\,y(0) - y'(0) - 4\mathcal{L}\{y\} = 0$ or $s^2\mathcal{L}\{y\} - s - 2 - 4\mathcal{L}\{y\} = 0$

$\Rightarrow \quad \mathcal{L}\{y\} = \dfrac{s+2}{s^2-4} = \dfrac{1}{s-2} \qquad \therefore \qquad y(t) = e^{2t}$

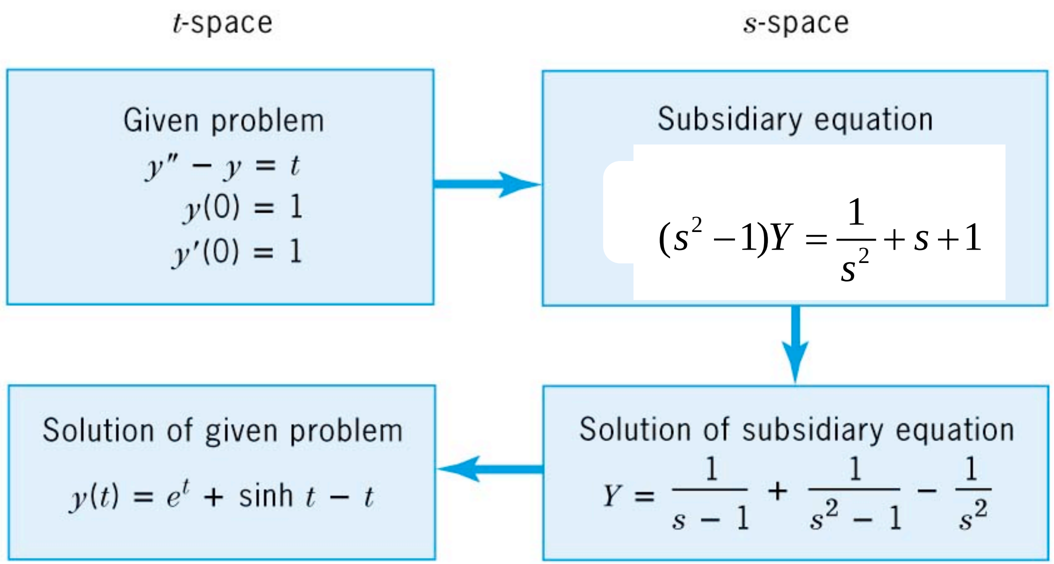

Fig. 108. Laplace transform method

[Exercise] $y'' + 4y = 0, \quad y(0) = 1, \; y'(0) = 2$ **(IVP!)**

$$\Rightarrow \quad y(t) = \cos 2t + \sin 2t$$

[Exercise] $y'' - 3y' + 2y = 4t - 6, \quad y(0) = 1, \; y'(0) = 3$ **(IVP!)**

$$(s^2 \bar{y} - s - 3) - 3(s\bar{y} - 1) + 2\bar{y} = \frac{4}{s^2} - \frac{6}{s}$$

$$\Rightarrow \quad \bar{y} = \frac{s^2 + 2s - 2}{s^2 (s-1)} = \frac{1}{s-1} + \frac{2}{s^2}$$

$$\therefore \quad y = \mathcal{L}^{-1}\left\{ \frac{s^2 + 2s - 2}{s^2 (s-1)} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{s-1} + \frac{2}{s^2} \right\} = e^t + 2t$$

[Exercise] $y'' - 5y' + 4y = e^{2t}, \quad y(0) = 1, \; y'(0) = 0$ **(IVP!)**

$$\Rightarrow \quad y(t) = -\frac{1}{2} e^{2t} + \frac{5}{3} e^t - \frac{1}{6} e^{4t}$$

Question: **Can a boundary-value problem (BVP) be solved by Laplace Transform method?**

[Example] $y'' + 9y = \cos 2t, \quad y(0) = 1, \quad y(\pi/2) = -1$

Let $y'(0) = c$

$$\therefore \quad \mathcal{L}\{ y'' + 9y \} = \mathcal{L}\{ \cos 2t \}$$

$$s^2 \bar{y} - s\, y(0) - y'(0) + 9\bar{y} = \frac{s}{s^2+4} \quad \text{or} \quad s^2 \bar{y} - s - c + 9\bar{y} = \frac{s}{s^2+4}$$

$$\therefore \quad \bar{y} = \frac{s+c}{s^2+9} + \frac{s}{(s^2+9)(s^2+4)} = \frac{4}{5}\frac{s}{s^2+9} + \frac{c}{s^2+9} + \frac{s}{5(s^2+4)}$$

$$\Rightarrow \quad y = \mathcal{L}^{-1}\{ \bar{y} \} = \frac{4}{5}\cos 3t + \frac{c}{3}\sin 3t + \frac{1}{5}\cos 2t$$

Now since $y(\pi/2) = -1$, we have $-1 = -c/3 - 1/5 \quad \Rightarrow \quad c = 12/5$

$$\Rightarrow \quad y = \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t + \frac{1}{5}\cos 2t$$

[Exercise] Find the general solution to $y'' + 9y = \cos 2t$ by Laplace Transform method.

Let

$y(0) = c_1$

$y'(0) = c_2$

Additional Remarks:

Since $\mathcal{L}\{ f'(t) \} = s\,\mathcal{L}\{ f(t) \} - f(0^+)$ if f(t) is continuous

If $f(0) = 0 \Rightarrow \mathcal{L}^{-1}\{ s\,\bar{f}(s) \} = f'(t)$ **(i.e., multiplied by s)**

[Example] If we know $\mathcal{L}^{-1}\left\{ \dfrac{1}{s^2+1} \right\} = \sin t$ then $\mathcal{L}^{-1}\left\{ \dfrac{s}{s^2+1} \right\} = ??$

[Solution] Since

$$\sin 0 = 0$$

$$\Rightarrow \quad \mathcal{L}^{-1}\left\{ \frac{s}{s^2+1} \right\} = \frac{d}{dt}\,\mathcal{L}^{-1}\left\{ \frac{1}{s^2+1} \right\} = \frac{d}{dt} \sin t = \cos t$$

(3) Laplace Transform of Integrals

If f(t) is piecewise continuous and $|\, f(t)\, | \le M\, e^{\gamma t}$, then

$$\mathcal{L}\left\{ \int_a^t f(\tau)\, d\tau \right\} = \frac{1}{s}\, \mathcal{L}\{ f(t) \} + \frac{1}{s} \int_a^0 f(\tau)\, d\tau$$

[Proof]

$$\mathcal{L}\left\{ \int_a^t f(\tau)\, d\tau \right\} = \int_0^\infty \left[\int_a^t f(\tau)\, d\tau \right] e^{-st}\, dt \qquad \text{(integration by parts)}$$

$$= \left[-\frac{e^{-st}}{s} \int_a^t f(\tau)\, d\tau \right]_0^\infty + \frac{1}{s} \int_0^\infty f(t)\, e^{-st}\, dt = \frac{1}{s} \int_a^0 f(\tau)\, d\tau + \frac{1}{s} \int_0^\infty f(t)\, e^{-st}\, dt = \frac{1}{s} \int_a^0 f(\tau)\, d\tau + \frac{1}{s}\, \mathcal{L}\{ f(t) \}$$

Special Cases: for a = 0,

$$\mathcal{L}\left\{ \int_0^t f(\tau)\, d\tau \right\} = \frac{1}{s}\, \mathcal{L}\{ f(t) \} = \frac{\bar{f}(s)}{s}$$

Inverse:

$$\mathcal{L}^{-1}\left\{ \frac{\bar{f}(s)}{s} \right\} = \int_0^t f(\tau)\, d\tau$$ **(divided by s!)**

[Example] If we know $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t$, then

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \int_0^t \frac{1}{2}\sin 2\tau\, d\tau = \frac{1-\cos 2t}{4}$$

[Exercise] If we know $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$, then $\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = ??$

$$\int_0^t\int_0^t\int_0^t \sin t\; dtdtdt$$

$$= \int_0^t\int_0^t (1-\cos t)dtdt = \int_0^t (t-\sin t)dt$$

$$= \left[\frac{t^2}{2}+\cos t\right]_0^t$$

$$= \frac{t^2}{2}+\cos t - 1$$

(4) Multiplication by t^n

$$\mathcal{L}\{ t^n f(t) \} = (-1)^n \frac{d^n \bar{f}(s)}{d s^n}$$

e.g., if n=1, then $\mathcal{L}\{ t f(t) \} = -\bar{f}'(s)$

[Proof]

$$\bar{f}(s) = \mathcal{L}\{ f(t) \} = \int_0^\infty e^{-st} f(t)\, dt$$

$$\frac{d\bar{f}(s)}{ds} = \frac{d}{ds} \int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty \left(\frac{\partial}{\partial s} e^{-st}\right) f(t)\, dt$$ (Leibniz formula)[1]

$$= \int_0^\infty -t\, e^{-st} f(t)\, dt = -\int_0^\infty t\, e^{-st} f(t)\, dt = -\mathcal{L}\{ t f(t) \}$$

$$\Rightarrow \quad \mathcal{L}\{ t f(t) \} = -\frac{d}{ds} \bar{f}(s) = -\frac{d}{ds} \mathcal{L}\{ f(t) \}$$

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1 Leibniz's Rule:

$$\frac{d}{d\alpha} \int_{\phi_1(\alpha)}^{\phi_2(\alpha)} F(x,\alpha)\, dx = \int_{\phi_1(\alpha)}^{\phi_2(\alpha)} \frac{\partial F}{\partial \alpha}\, dx + F(\phi_2, \alpha) \frac{d\phi_2}{d\alpha} - F(\phi_1, \alpha) \frac{d\phi_1}{d\alpha}$$

[Example] $\mathcal{L}\{ e^{2t} \} = \dfrac{1}{s-2}$

$$\mathcal{L}\{ t\, e^{2t} \} = -\frac{d}{ds}\left(\frac{1}{s-2}\right) = \frac{1}{(s-2)^2}$$

$$\mathcal{L}\{ t^2 e^{2t} \} = \frac{d^2}{ds^2}\left(\frac{1}{s-2}\right) = \frac{2}{(s-2)^3}$$

[Exercise] $\mathcal{L}\{ t \sin \omega t \} = ??$ $\mathcal{L}\{ t^2 \cos \omega t \} = ??$

[Example] $t\, y'' - t\, y' - y = 0, \quad y(0) = 0, \; y'(0) = 3$

[Solution] Taking Laplace transforms of both sides of the differential equation, we have

$$\mathcal{L}\{ t\, y'' - t\, y' - y \} = \mathcal{L}\{ 0 \} \qquad \text{or} \qquad \mathcal{L}\{ t\, y'' \} - \mathcal{L}\{ t\, y' \} - \mathcal{L}\{ y \} = 0$$

Note that $\mathcal{L}\{ t\, y'' \} = -\dfrac{d}{ds}\mathcal{L}\{ y'' \} = -\dfrac{d}{ds}\left(s^2 \bar{y} - s\, y(0) - y'(0) \right) = -s^2 \bar{y}\,' - 2 s \bar{y} + y(0) = -s^2 \dfrac{d\bar{y}}{ds} - 2s\bar{y}$

$$\mathcal{L}\{ t\, y' \} = -\frac{d}{ds}\mathcal{L}\{ y' \} = -\frac{d}{ds}\left(s \bar{y} - y(0) \right) = -s\bar{y}\,' - \bar{y} = -s\frac{d\bar{y}}{ds} - \bar{y}$$

$$\mathcal{L}\{ y \} = \bar{y}$$

$$\Rightarrow \quad -s^2\bar{y}\,' - 2s\bar{y} + s\bar{y}\,' + \bar{y} - \bar{y} = 0 \quad \text{or} \quad \bar{y}\,' + \frac{2}{s-1}\bar{y} = 0 \quad \Rightarrow \frac{d\bar{y}}{\bar{y}} = -\frac{2}{s-2}ds$$

Solve the above equation by separation of variable for $\bar{y}$, we have

$$\bar{y} = \frac{c}{(s-1)^2} \qquad \text{or} \qquad y = c\,t\,e^t$$

But $y'(0) = 3$, we have $3 = y'(0) = c\,(t+1)\,e^t \Big|_{t=0} = c \quad \Rightarrow \quad y(t) = 3\,t\,e^t$

[Example] Evaluate $\mathcal{L}^{-1}\left\{\tan^{-1}\left(\frac{1}{s}\right)\right\}$ *indirectly* by (4), i.e., $\mathcal{L}\{\, t\, f(t)\,\} = -F'(s)$ and $F(s) = \tan^{-1}(1/s)$.

[Solution] It is easier to evaluate the inversion of the derivative of $\tan^{-1}\left(\frac{1}{s}\right)$ wrt s.

By making use of the *identity*: $(\tan^{-1} s)' = +\frac{1}{s^2+1}$, one can obtain $F'(s) = (\tan^{-1}(1/s))' = \frac{-1/s^2}{(1/s)^2+1} = -\frac{1}{s^2+1}$

$$L^{-1}\left\{\frac{d}{ds}\tan^{-1}\left(\frac{1}{s}\right)\right\} = \mathcal{L}^{-1}\left\{\frac{-1}{s^2+1}\right\} = -\sin t$$

$$-\sin t = L^{-1}\left\{\frac{d}{ds}\tan^{-1}\left(\frac{1}{s}\right)\right\} = \mathcal{L}^{-1}\{F'(s)\} = -t\,f(t) = -tL^{-1}\left\{\tan^{-1}\left(\frac{1}{s}\right)\right\},$$

$$\therefore -\sin t = -t\,f(t).$$

$$\Rightarrow \qquad L^{-1}\left\{\tan^{-1}\left(\frac{1}{s}\right)\right\} = f(t) = \frac{\sin t}{t}$$

[Example] Evaluate $\mathcal{L}^{-1}\left\{ \ln\left(1 + \frac{1}{s}\right) \right\}$ *indirectly* by (4)

$$\mathcal{L}^{-1}\left\{ \ln\left(1 + \frac{1}{s}\right) \right\} = \mathcal{L}^{-1}\{ \bar{f}(s) \} = f(t) \quad \text{and} \quad \bar{f}\,'(s) = \frac{d}{ds}\left(\ln\left(1 + \frac{1}{s}\right)\right) = -\frac{1}{s} + \frac{1}{s+1}$$

Since from (4) we have $\mathcal{L}^{-1}\{ \bar{f}\,'(s) \} = -t\, f(t)$ $\Rightarrow$ $-1 + e^{-t} = -t\, f(t)$

$$\therefore \quad f(t) = \frac{1 - e^{-t}}{t}$$

(5) Division by t

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(\tilde{s})\, d\tilde{s} \qquad \text{provided that } \frac{f(t)}{t} \text{ exists for } t \to 0.$$

[Example] It is known that

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1} \qquad \text{and} \qquad \lim_{t\to 0} \frac{\sin t}{t} = 1$$

$$\Rightarrow \quad \mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{d\tilde{s}}{\tilde{s}^2+1} = -\tan^{-1}\left(\frac{1}{s}\right)\Bigg|_s^\infty = \tan^{-1}\left(\frac{1}{s}\right)$$

[Example] Determine the Laplace Transform of $\frac{\sin^2 t}{t}$.

$$\mathcal{L}\{\sin^2 t\} = \mathcal{L}\left\{\frac{1-\cos 2t}{2}\right\} = \frac{1}{2s} - \frac{1}{2}\frac{s}{s^2+4} = \frac{2}{s(s^2+4)}$$

$$\text{Thus, } \mathcal{L}\left\{\frac{\sin^2 t}{t}\right\} = \int_s^\infty \frac{2}{s(s^2+4)}\,ds = \int_s^\infty \frac{1}{2s} - \frac{1}{2}\frac{s}{s^2+4}\,ds$$

$$= \left[\frac{1}{2}\ln s - \frac{1}{4}\ln(s^2+4)\right]_s^\infty = \left[\frac{1}{4}\ln\frac{s^2}{s^2+4}\right]_s^\infty = \frac{1}{4}\ln\frac{s^2+4}{s^2} \qquad \because \lim_{s\to\infty}\left(\ln\frac{s^2}{s^2+4}\right)=\ln(1)=0$$

[Example] Evaluate the integral $\int_0^\infty e^{-t}\frac{\sin^2 t}{t}\,dt$

$$\mathcal{L}\left\{\frac{\sin^2 t}{t}\right\} = \int_0^\infty e^{-st}\frac{\sin^2 t}{t}\,dt \quad \text{as } \boxed{s=1},$$

$$\text{Thus, } \int_0^\infty e^{-t}\frac{\sin^2 t}{t}\,dt = \frac{1}{4}\ln\frac{s^2+4}{s^2}\Big|_{s=1} = \frac{1}{4}\ln 5$$

(6) First Translation or Shifting Property (s-Shifting)

If $\mathcal{L}\{ f(t) \} = \bar{f}(s)$, then $\mathcal{L}\{ e^{at} f(t) \} = \bar{f}(s-a)$

If $\mathcal{L}^{-1}\{ \bar{f}(s) \} = f(t)$, then $\mathcal{L}^{-1}\{ \bar{f}(s - a) \} = e^{at} f(t)$

[Example] $\mathcal{L}\{ \cos 2t \} = \dfrac{s}{s^2 + 4}$

$$\mathcal{L}\{ e^{-t} \cos 2t \} = \frac{s+1}{(s+1)^2 + 4} = \frac{s+1}{s^2 + 2s + 5}$$

[Exercise] $\mathcal{L}\{ e^{-2t} \sin 4t \}$

[Example] $\mathcal{L}^{-1}\left\{ \dfrac{6s - 4}{s^2 - 4s + 20} \right\} = \mathcal{L}^{-1}\left\{ \dfrac{6s-4}{(s-2)^2 + 16} \right\}$

$$= \mathcal{L}^{-1}\left\{ \frac{6(s-2) + 8}{(s-2)^2 + 16} \right\} = 6L^{-1}\left\{ \frac{s-2}{(s-2)^2 + 4^2} \right\} + 2L^{-1}\left\{ \frac{4}{(s-2)^2 + 4^2} \right\}$$

$$= 6e^{2t} \cos 4t + 2e^{2t} \sin 4t = 2e^{2t}(3 \cos 4t + \sin 4t)$$

(7) Second Translation or Shifting Property (**t-Shifting**)

If $\mathcal{L}\{ f(t) \} = \bar{f}(s)$ and $g(t) = \begin{cases} f(t-a) & \text{if} \quad t > a \\ 0 & \text{if} \quad t < a \end{cases}$

$$\Rightarrow \quad \mathcal{L}\{ g(t) \} = e^{-as} \bar{f}(s)$$

[Example] $\mathcal{L}\{ t^3 \} = \dfrac{3!}{s^4} = \dfrac{6}{s^4}$

$$g(t) = \begin{cases} (t-2)^3 & t > 2 \\ 0 & t < 2 \end{cases}$$

$$\Rightarrow \quad \mathcal{L}\{ g(t) \} = \frac{6}{s^4} e^{-2s}$$

(8) Step Functions, Impulse Functions and Periodic Functions

(a) Unit Step Function (Heaviside Function) u(t–a)

Definition:

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$$

Thus, the function $g(t) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$

can be written as $g(t) = f(t-a)\, u(t-a)$

The Laplace transform of g(t) can be calculated as

$$\mathcal{L}\{ f(t-a)\, u(t-a) \} = \int_0^\infty e^{-st} f(t-a)\, u(t-a)\, dt$$

$$= \int_a^\infty e^{-st} f(t-a)\, dt \qquad (\text{ by letting } x = t-a)$$

$$= \int_0^\infty e^{-s(x+a)} f(x)\, dx \quad = e^{-sa} \int_0^\infty e^{-sx} f(x)\, dx = e^{-sa} \mathcal{L}\{ f(t) \} = e^{-sa}\, \bar{f}(s)$$

$$\Rightarrow \quad \mathcal{L}\{ f(t-a)\, u(t-a) \} = e^{-as} \mathcal{L}\{ f(t) \} = e^{-as}\, \bar{f}(s) \quad \text{and} \quad \mathcal{L}^{-1}\{ e^{-sa}\, \bar{f}(s) \} = f(t-a)\, u(t-a)$$

[Example] $\mathcal{L}\{ \sin a(t-b)\, u(t-b) \} = e^{-bs} \mathcal{L}\{ \sin at \} = \dfrac{a\, e^{-bs}}{s^2 + a^2}$

[Example] $\mathcal{L}\{ u(t-a) \} = \dfrac{e^{-as}}{s}$

[Example] Calculate $\mathcal{L}\{ f(t) \}$

$$\text{where } f(t) = \begin{cases} e^t & 0 \le t \le 2\pi \\ e^t + \cos t & t > 2\pi \end{cases}$$

[Solution]

Since the function

$$u(t-2\pi)\cos(t-2\pi) = \begin{cases} 0 & t < 2\pi \\ \cos(t-2\pi)\ (= \cos t) & t > 2\pi \end{cases}$$

$\therefore$ the function f(t) can be written as

$$f(t) = e^t + u(t-2\pi)\cos(t-2\pi)$$

$$\Rightarrow \quad \mathcal{L}\{ f(t) \} = \mathcal{L}\{ e^t \} + \mathcal{L}\{ u(t-2\pi)\cos(t-2\pi) \} = \frac{1}{s-1} + \frac{s\, e^{-2\pi s}}{1 + s^2}$$

[Example] $\mathcal{L}^{-1}\left\{ \frac{1 - e^{-\pi s/2}}{1 + s^2} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{s^2 + 1} \right\} - \mathcal{L}^{-1}\left\{ \frac{e^{-\pi s/2}}{s^2 + 1} \right\}$

$$= \sin t - u\left(t - \frac{\pi}{2} \right) \sin\left(t - \frac{\pi}{2} \right) = \sin t + u\left(t - \frac{\pi}{2} \right) \cos t$$

[Example] Rectangular Pulse

$$f(t) = u(t-a) - u(t-b)$$

$$\mathcal{L}\{ f(t) \} = \mathcal{L}\{ u(t-a) \} - \mathcal{L}\{ u(t-b) \} = \frac{e^{-as}}{s} - \frac{e^{-bs}}{s}$$

[Example] Staircase

$$f(t) = u(t-a) + u(t-2a) + u(t-3a) + \ldots$$

$$\mathcal{L}\{ f(t) \} = \mathcal{L}\{ u(t-a) \} + \mathcal{L}\{ u(t-2a) \} + \mathcal{L}\{ u(t-3a) \} + \ldots$$

$$= \frac{1}{s} \left(e^{-as} + e^{-2as} + e^{-3as} + \ldots \right)$$

If $as > 0$, $e^{-as} < 1$, and that

$$1 + x + x^2 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1 - x} \quad , \ | x | < 1$$

then, for $s > 0$, $\mathcal{L}\{ f(t) \} = \frac{1}{s} \frac{e^{-as}}{1 - e^{-as}}$

[Example] Square Wave

$$f(t) = u(t) - 2\,u(t-a) + 2\,u(t-2a) - 2\,u(t-3a) + \ldots \quad \Rightarrow \quad \mathcal{L}\{\,f(t)\,\} = \frac{1}{s}\left(1 - 2e^{-as} + 2e^{-2as} - 2e^{-3as} + \ldots\right)$$

$$= \frac{1}{s}\left\{2\left(1 - e^{-as} + e^{-2as} - e^{-3as} + \ldots\right) - 1\right\} = \frac{1}{s}\left\{\frac{2}{1+e^{-as}} - 1\right\} = \frac{1}{s}\left[\frac{1-e^{-as}}{1+e^{-as}}\right]$$

$$= \frac{1}{s}\left[\frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}}\right] = \frac{1}{s}\tanh\left(\frac{a\,s}{2}\right)$$

[Example] Solve $\begin{cases} y' + 2\,y + 6\displaystyle\int_0^t z\,dt = -2\,u(t) \\ y' + z' + z = 0 \end{cases}$ with $y(0) = -5, \; z(0) = 6$

[Solution] We take the Laplace transform of the above set of equations:

$$\begin{cases} (\,s\,L\{\,y\,\} + 5\,) + 2\,L\{\,y\,\} + \dfrac{6}{s}\,L\{\,z\,\} = -\dfrac{2}{s} \\ (\,s\,L\{\,y\,\} + 5\,) + (\,s\,L\{\,z\,\} - 6\,) + L\{\,z\,\} = 0 \end{cases} \quad \text{or} \quad \begin{cases} (s^2 + 2s)\,\bar{y} + 6\,\bar{z} = -2 - 5\,s \\ s\,\bar{y} + (s+1)\,\bar{z} = 1 \end{cases}$$

$$\bar{y} = \frac{-5\,s^2 - 7\,s - 8}{s^3 + 3\,s^2 - 4\,s} = \frac{2}{s} - \frac{4}{s-1} - \frac{3}{s+4}$$

$$\bar{z} = \frac{1 - s\bar{y}}{s+1} = \frac{2(3s+2)}{(s-1)(s+4)} = \frac{2}{s-1} + \frac{4}{s+4}$$

$$\Rightarrow \quad y = \mathcal{L}^{-1}\{\,\bar{y}\,\} = 2\,u(t) - 4\,e^{t} - 3\,e^{-4t}$$

$$z = 2e^{t} + 4e^{-4t}$$

[Exercise]

$$\begin{cases} y' + y + 2z' + 3z = e^{-t} \\ 3y' - y + 4z' + z = 0 \end{cases}$$

$$y(0) = -1, \quad z(0) = 0$$

[Exercise] $y'' + y = f(t), \quad y(0) = y'(0) = 0$

$$\text{where } f(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & t > 1 \end{cases}$$

(b) Unit Impulse Function (Dirac Delta Function) $\delta(t-a)$

Definition:

Let $f_k(t) = \begin{cases} 1/k & a \le t \le a+k \\ 0 & \text{otherwise} \end{cases}$

and $I_k = \int_0^\infty f_k(t)\,dt = 1$

Define: $\delta(t-a) = \lim_{k\to 0} f_k(t)$

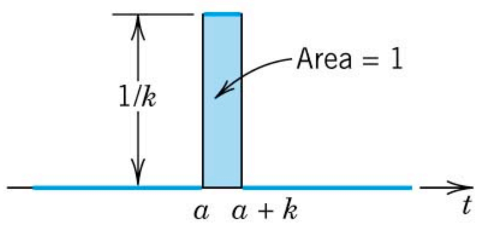

Fig. 117. The function $f_k(t-a)$ in (5)

From the definition, we know

$$\delta(t-a) = \begin{cases} \infty & t = a \\ 0 & t \neq a \end{cases}$$

and $\int_0^\infty \delta(t-a)\,dt = 1$ $\qquad \int_{-\infty}^\infty \delta(t-a)\,dt = 1$

Note that

$$\int_0^\infty \delta(t)\,dt = 1$$

$$\int_0^\infty \delta(t)\,g(t)\,dt = g(0) \text{ for any continuous function } g(t)$$

$$\int_0^\infty \delta(t-a)\,g(t)\,dt = g(a)$$

The Laplace transform of $\delta(t)$ is

$$\mathcal{L}\{\,\delta(t-b)\,\} = \int_0^\infty e^{-st}\,\delta(t-b)\,dt = e^{-bs}$$

[Question] $\mathcal{L}\{\,e^t \cos t\ \delta(t-3)\,\} = ??$

[Example] Find the solution of y for

$$y'' + 2y' + y = \delta(t-1), \quad y(0) = 2, \quad y'(0) = 3$$

[Solution]

The Laplace transform of the above equation is

$$(s^2 \bar{y} - 2s - 3) + 2(s\bar{y} - 2) + \bar{y} = e^{-s}$$

or $$\bar{y} = \frac{2s + 7 + e^{-s}}{s^2 + 2s + 1} = \frac{2(s+1)}{(s+1)^2} + \frac{5}{(s+1)^2} + \frac{e^{-s}}{(s+1)^2}$$

$$= \frac{2}{s+1} + \frac{5}{(s+1)^2} + \frac{e^{-s}}{(s+1)^2}$$

Since

$$\mathcal{L}\{t e^{-t}\} = \frac{1}{(s+1)^2} \qquad \left(\text{Recall } \mathcal{L}\{t\} = \frac{1}{s^2}\right)$$

$$\Rightarrow \quad \mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s+1)^2}\right\} = (t-1)\, e^{-(t-1)}\, u(t-1)$$

$$\therefore \quad y = 2e^{-t} + 5t e^{-t} + (t-1)\, e^{-(t-1)}\, u(t-1)$$

$$= e^{-t}\,[2 + 5t + e\,(t-1)\, u(t-1)]$$

(c) Periodic Functions

For all t, $f(t+p) = f(t)$, then f(t) is said to be ***periodic function* with period p**.

Theorem:

The Laplace transform of a piecewise continuous periodic function f(t) with period p is

$$\mathcal{L}\{f\} = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t)\, dt$$

[Proof]

$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^p e^{-st} f(t)\, dt + \int_p^{2p} e^{-st} f(t)\, dt$$

$$+ \int_{2p}^{3p} e^{-st} f(t)\, dt + \ldots$$

$$\int_{kp}^{(k+1)p} e^{-st} f(t)\, dt = \int_0^p e^{-s(u+kp)} f(u+kp)\, du$$ …which is the (k+1)th integral!

(where $u = t - kp$ and $0<u<p$)

$$= e^{-skp} \boxed{\int_0^p e^{-su} f(u)\, du}$$ ...since $f(u+kp) = f(u)$

$$\therefore \qquad \mathcal{L}\{ f \} = \sum_{k=0}^{\infty} e^{-skp} \int_0^p e^{-su} f(u)\, du$$

$$= \left[\int_0^p e^{-su} f(u)\, du \right] \sum_{k=0}^{\infty} (e^{-sp})^k$$

$$= \frac{\int_0^p e^{-su} f(u)\, du}{1 - e^{-ps}}$$

[Example] Find $\mathcal{L}\{ | \sin at | \}$, $a > 0$

[Solution] $p = \dfrac{\pi}{a}$ (due to $|\bullet|$)

$$\mathcal{L}\{ | \sin at | \} = \frac{\int_0^p e^{-su} f(t)\, dt}{1 - e^{-ps}}$$

$$= \frac{\int_0^{\pi/a} e^{-st} \sin at\, dt}{1 - e^{-\pi s/a}}$$ (Use integration by parts twice)

$$= \frac{a}{s^2 + a^2} \frac{1 + e^{-\pi s/a}}{1 - e^{-\pi s/a}} = \frac{a}{s^2 + a^2} \frac{\left(e^{\frac{\pi s}{2a}} + e^{-\frac{\pi s}{2a}} \right)/2}{\left(e^{\frac{\pi s}{2a}} - e^{-\frac{\pi s}{2a}} \right)/2}$$

$$= \frac{a}{s^2 + a^2} \coth\left(\frac{\pi s}{2a}\right)$$

[Example] $y'' + 2y' + 5y = f(t), \quad y(0) = y'(0) = 0$

where $f(t) = u(t) - 2u(t-\pi) + 2u(t-2\pi) - 2u(t-3\pi) + \ldots$ **(Square Wave!)**

[Solution]

The Laplace transform of the square wave f(t) is

$$\mathcal{L}\{ f(t) \} = \frac{1}{s} \frac{1 - e^{-\pi s}}{1 + e^{-\pi s}}$$ **(This was derived previously!)**

$$\Rightarrow \quad s^2 \bar{y} + 2s\bar{y} + 5\bar{y} = \frac{1}{s} \frac{1 - e^{-\pi s}}{1 + e^{-\pi s}}$$

or $$\bar{y} = \frac{1}{s^2 + 2s + 5} \frac{1}{s} \frac{1 - e^{-\pi s}}{1 + e^{-\pi s}}$$

Now $$\frac{1}{s(s^2 + 2s + 5)}$$

$$= \frac{1}{5}\left[\frac{1}{s} - \frac{s+2}{s^2 + 2s + 5} \right] = \frac{1}{5}\left[\frac{1}{s} - \frac{s+2}{(s+1)^2 + 2^2} \right]$$

$$= \frac{1}{5}\left[\frac{1}{s} - \frac{(s+1)}{(s+1)^2+2^2} - \frac{1}{2}\frac{2}{(s+1)^2+2^2}\right]$$

and $\dfrac{1 - e^{-\pi s}}{1 + e^{-\pi s}} = (1 - e^{-\pi s})(1 - e^{-\pi s} + e^{-2\pi s} - e^{-3\pi s} + \ldots)$

$= 1 - 2\,e^{-\pi s} + 2\,e^{-2\pi s} - 2\,e^{-3\pi s} + \ldots$ **(derived previously)**

$$\Rightarrow \quad \bar{y} = \frac{1}{5}\left[\frac{1}{s} - \frac{s+2}{(s+1)^2+2^2}\right](1 - 2\,e^{-\pi s} + 2\,e^{-2\pi s} - 2\,e^{-3\pi s} + \ )$$

The inverse Laplace transform of $\bar{y}$ can be calculated in the following way:

$$\mathcal{L}^{-1}\left\{\frac{1}{5}\left[\frac{1}{s} - \frac{s+2}{(s+1)^2+2^2}\right]\right\} = L^{-1}\left\{\frac{1}{5}\left[\frac{1}{s} - \frac{(s+1)}{(s+1)^2+2^2} - \frac{1}{2}\frac{2}{(s+1)^2+2^2}\right]\right\}$$

$$= \frac{1}{5}\left[1 - g(t)\right] = \frac{1}{5}\left[1 - e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)\right]$$ (k=0, the first term)

$$\mathcal{L}^{-1}\left\{\frac{2}{5}\left[\frac{1}{s} - \frac{s+2}{(s+1)^2+2^2}\right]e^{-k\pi s}\right\}$$ (k=1, 2, 3, …)

$$= \frac{2}{5}\left(1 - g(t-k\pi)\right)\ u(t-k\pi)$$

But $g(t-k\pi) = e^{-(t-k\pi)}\left(\cos 2(t-k\pi) + \dfrac{1}{2}\sin 2(t-k\pi)\right)$

$$= e^{k\pi}\,g(t) = e^{k\pi}\left[e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)\right]$$

$$\therefore \quad y(t) = \frac{1}{5}(1 - g(t)) - \frac{2}{5}(1 - e^{\pi}g(t))\ u(t-\pi)$$

$$+ \frac{2}{5}(1 - e^{2\pi}g(t))\ u(t-2\pi) - \frac{2}{5}(1 - e^{3\pi}g(t))\ u(t-3\pi)$$

$$+ \ldots\ldots$$

$$= \frac{1}{5}(1 - 2u(t-\pi) + 2u(t-2\pi) - 2u(t-3\pi) + \ldots)$$

$$- \frac{g(t)}{5}(1 - 2e^{\pi}u(t-\pi) + 2e^{2\pi}u(t-2\pi) - \ldots)$$

$$= \frac{1}{5}\Big(f(t) - g(t)(1 - 2e^{\pi}u(t-\pi) + 2e^{2\pi}u(t-2\pi)$$

$$- 2e^{3\pi}u(t-3\pi) + \ldots)\Big)$$

(9) Change of Scale Property

$$\mathcal{L}\{ f(t) \} = \bar{f}(s)$$

then $$\mathcal{L}\{ f(at) \} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$$

[Proof]

$$\mathcal{L}\{ f(at) \} = \int_0^\infty e^{-st} f(at)\, dt = \int_0^\infty e^{-su/a} f(u)\, d(u/a)$$

$$= \frac{1}{a}\int_0^\infty e^{-su/a} f(u)\, du = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$$

[Exercise] Given that $\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \tan^{-1}(1/s)$

Find $\mathcal{L}\left\{\frac{\sin at}{t}\right\} = ??$

Note that $\mathcal{L}\left\{\frac{\sin at}{at}\right\} = \frac{1}{a}\bar{f}(s/a) = \frac{1}{a}\tan^{-1}(a/s)$

$$\Rightarrow \quad \mathcal{L}\left\{\frac{\sin at}{t}\right\} = a\,\mathcal{L}\left\{\frac{\sin at}{at}\right\} = \tan^{-1}(a/s)$$

(10) Laplace Transform of Convolution Integrals

Definition of Convolution

If f and g are piecewise continuous functions, then the **convolution of f and g**, written as (f*g), is defined by

$$(f*g)(t) \equiv \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

Properties

(a) f*g = g*f (commutative law)

$$(f*g)(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

$$= -\int_t^0 f(v)\, g(t-v)\, dv \qquad \text{(by letting } v = t - \tau\text{)}$$

$$= \int_0^t g(t-v)\, f(v)\, dv = (g*f)(t) \qquad \text{q.e.d.}$$

(b) $f*(g_1 + g_2) = f*g_1 + f*g_2$ (linearity)

(c) $(f*g)*v = f*(g*v)$

(d) $f*0 = 0*f = 0$

(e) 1*f ≠ f in general

<u>Convolution Theorem</u>

Let $\overline{f}(s) = \mathcal{L}\{ f(t) \}$ and $\overline{g}(s) = \mathcal{L}\{ g(t) \}$

then $\mathcal{L}\{ (f*g)(t) \} = \overline{f}(s)\, \overline{g}(s)$

[Proof]

$$\bar{f}(s)\,\bar{g}(s) = \left[\int_0^\infty e^{-s\tau} f(\tau)\,d\tau\right]\left[\int_0^\infty e^{-sv} g(v)\,dv\right]$$

$$= \int_0^\infty \int_0^\infty e^{-s(\tau+v)} f(\tau)\, g(v)\, dv\, d\tau$$

Let $t = \tau + v$ and **consider inner integral with <u>τ fixed (given)</u>**, then

$dt = dv$ and

$$\bar{f}(s)\,\bar{g}(s) = \int_0^\infty \int_\tau^\infty e^{-st} f(\tau)\, g(t-\tau)\, dt\, d\tau$$

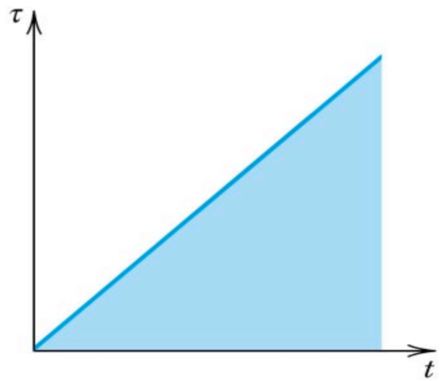

Fig. 123. Region of integration in the $t\tau$-plane in the proof of Theorem 1

$$\int_0^\infty \int_\tau^\infty \_\_\_\, dt\, d\tau = \int_0^\infty \int_0^t \_\_\_\, d\tau\, dt$$

$$\Rightarrow \quad \bar{f}(s)\,\bar{g}(s) = \int_0^\infty \int_\tau^\infty e^{-st} f(\tau)\, g(t-\tau)\, dt\, d\tau$$

$$= \int_0^\infty \int_0^t e^{-st} f(\tau)\, g(t-\tau)\, d\tau\, dt$$

$$= \int_0^\infty e^{-st} \left[\int_0^t g(t-\tau)\, f(\tau)\, d\tau \right] dt$$

$$= \int_0^\infty e^{-st} (g*f)(t)\, dt = \int_0^\infty e^{-st} (f*g)(t)\, dt$$

$$= \mathcal{L}\{ f*g \}$$

Corollary

If $\bar{f}(s) = \mathcal{L}\{ f(t) \}$ and $\bar{g}(s) = \mathcal{L}\{ g(t) \}$, then

$$\mathcal{L}^{-1}\{ \bar{f}(s)\, \bar{g}(s) \} = (f*g)(t)$$

[Example] Find $\mathcal{L}^{-1}\left\{ \dfrac{s}{(s^2+1)^2} \right\}$

Recall that the Laplace transforms of cos t and sin t are

$$\mathcal{L}\{\cos t\} = \frac{s}{s^2+1} \qquad \mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$$

Thus, $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\,\frac{1}{s^2+1}\right\}$$

$$= \sin t * \cos t$$

Since $\sin t * \cos t = \int_0^t \sin(t-\tau)\cos\tau \, d\tau$

$$= \int_0^t (\sin t \cos\tau - \cos t \sin\tau)\, \cos\tau\, d\tau$$

$$= \sin t \int_0^t \cos^2\tau\, d\tau - \cos t \int_0^t \sin\tau \cos\tau\, d\tau$$

$$= \frac{1}{2}\left[\sin t\left(t + \frac{1}{2}\sin 2t\right) + \cos t\left(\frac{\cos 2t - 1}{2}\right)\right]$$

$$= \frac{t \sin t}{2}$$

Note, cos(A-B)=cos A cos B + sin A sin B

[Example] Find the solution of y to the differential equation

$$y'' + y = f(t), \qquad y(0) = 0, \qquad y'(0) = 1$$

and $$f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & t > 1 \end{cases}$$

[Solution]

The function f(t) can be written in terms of unit step functions:

$$f(t) = u(t) - u(t-1)$$

Now take the Laplace transforms on both sides of the differential equation, we have

$$s^2 \bar{y} - 1 + \bar{y} = \frac{1 - e^{-s}}{s}$$

or $$\bar{y} = \frac{1 + s - e^{-e}}{s\,(s^2 + 1)} = \frac{1}{s} - \frac{s - 1}{s^2 + 1} - \frac{e^{-s}}{s} \frac{1}{s^2 + 1}$$

$$\therefore \quad y = 1 - \cos t + \sin t - [\sin t * u(t-1)]$$

But the convolution $\sin t * u(t-1) = \int_0^t \sin(t-\tau)\, u(\tau-1)\, d\tau$

For $t < 1$, $u(t-1) = 0$, $\sin t * u(t-1) = 0$

and for $t > 1$, $u(t-1) = 1$,

$$\int_0^t \sin(t-\tau)\, u(\tau-1)\, d\tau = \int_1^t \sin(t-\tau)\, d\tau$$

Thus, $\sin t * u(t-1) = u(t-1) \int_1^t \sin(t-\tau)\, d\tau$

$$= u(t-1) \cos(t-\tau) \Big|_1^t = u(t-1)\, [1 - \cos(t-1)]$$

$$\Rightarrow \quad y = 1 - \cos t + \sin t - u(t-1)\, [1 - \cos(t-1)]$$

[Example] Volterra Integral Equation

$$y(t) = f(t) + \int_0^t g(t-\tau)\, y(\tau)\, d\tau$$

where f(t) and g(t) are continuous.

The solution of y can easily be obtained by taking Laplace transforms of the above integral equation:

$$\bar{y}(s) = \bar{f}(s) + \bar{g}(s)\,\bar{y}(s)$$

$$\Rightarrow \quad \bar{y}(s) = \frac{\bar{f}(s)}{1 - \bar{g}(s)}$$

For example, to solve

$$y(t) = t^2 + \int_0^t \sin(t-\tau)\, y(\tau)\, d\tau$$

$$\Rightarrow \quad \bar{y} = \frac{2}{s^3} + \frac{1}{s^2+1}\bar{y}$$

$$\text{or} \quad \bar{y} = \frac{2}{s^3} + \frac{2}{s^5} \qquad \because L\left\{t^n\right\} = \frac{n!}{s^{n+1}}$$

$$\Rightarrow \quad y = t^2 + \frac{1}{12} t^4$$

(11) Limiting Values

(a) Initial-Value Theorem

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} \mathbf{s}\,\bar{f}(s)$$

(b) Final-Value Theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} \mathbf{s}\,\bar{f}(s)$$

[Example] $f(t) = 3\,e^{-2t}$, $f(0) = 3$, $f(\infty) = 0$

$$\bar{f}(s) = \mathcal{L}\{\,f(t)\,\} = \frac{3}{s+2}$$

$$\lim_{s\to\infty} s\,\bar{f}(s) = \frac{3s}{s+2} = 3 \Rightarrow \quad f(0)$$

$$\lim_{s\to 0} s\,\bar{f}(s) = \frac{3s}{s+2} = 0 \Rightarrow \quad f(\infty)$$

[Exercise] Prove the above theorems

$$\mathcal{L}\{ f'(t) \} = s\,\mathcal{L}\{ f(t) \} - f(0^+)$$

3 Partial Fractions

- Please read Sec. 5.6 of the Textbook

$$\mathcal{L}^{-1}\left\{ \frac{F(s)}{G(s)} \right\} = ??$$

where F(s) and G(s) are polynomials in s.

Case 1 G(s) = 0 has distinct real roots
(i.e., G(s) contains unrepeated factors (s – a))

Case 2 . . .

. . .

4 Laplace Transforms of Some Special Functions

(1) Error Function

Definition:

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2}\, dx$$ Error Function

$$\operatorname{erfc}(t) \equiv 1 - \operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-x^2}\, dx$$ Complementary Error Function

[Example] Find $\mathcal{L}\{ \operatorname{erf}\sqrt{t} \}$

$$\operatorname{erf}\sqrt{t} = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-x^2}\, dx = \frac{1}{\sqrt{\pi}} \int_0^t u^{-1/2} e^{-u}\, du$$

(by letting $u = x^2$)

$$\therefore \quad \mathcal{L}\{ \operatorname{erf}\sqrt{t} \} = \frac{1}{\sqrt{\pi}} \mathcal{L}\left\{ \int_0^t u^{-1/2} e^{-u}\, du \right\}$$

$$\left(\text{Recall that } \mathcal{L}\left\{ \int_0^t f(\tau)\, d\tau \right\} = \frac{1}{s} \mathcal{L}\{ f(t) \} \right)$$

$$\Rightarrow \quad \mathcal{L}\{ \operatorname{erf}\sqrt{t} \} = \frac{1}{\sqrt{\pi}} \frac{1}{s} \mathcal{L}\{ t^{-1/2} e^{-t} \}$$

But $\mathcal{L}\{ t^{-1/2} \} = \dfrac{\Gamma(1/2)}{\sqrt{s}} = \dfrac{\sqrt{\pi}}{\sqrt{s}}$ $\left(\because L\{t^{\alpha}\} = \dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}} \right)$

we have $\mathcal{L}\{ t^{-1/2} e^{-t} \} = \dfrac{\sqrt{\pi}}{\sqrt{s+1}}$ (by s-shift)

$\Rightarrow$ $\mathcal{L}\{ \operatorname{erf} \sqrt{t} \} = \dfrac{1}{s\sqrt{s+1}}$

[Exercise] Find $\mathcal{L}^{-1}\left\{ \dfrac{1}{\sqrt{s}\,(s-1)} \right\} = ?? \Rightarrow e^{t} \operatorname{erf} \sqrt{t}$

(2) Bessel Functions

[Example] Find $\mathcal{L}\{ J_0(t) \}$

Note that

$$t^2 \frac{d^2 y}{dt^2} + t\frac{dy}{dt} + \left(t^2 - p^2\right) y = 0$$

$$\frac{d}{dt}\left[t^{-p} J_p(t)\right] = -t^{-p} J_{p+1}(t)$$

[Solution]

Note that $J_0(t)$ satisfies the Bessel's differential equation:

$t J_0''(t) + J_0'(t) + t J_0(t) = 0$ (p=0)

We now take $\mathcal{L}$ on both sides and note that

$J_0(0) = 1$ and $J_0'(0) = -J_1(0) = 0$

$$\Rightarrow \quad -\frac{d}{ds}\left(s^2 \bar{J}_0 - s\,(1) - 0 \right) + (s\,\bar{J}_o - 1) - \frac{d\bar{J}_0}{ds} = 0$$

$$\therefore \left(s^2+1\right)\bar{J}_0' + s\bar{J}_0 = 0 \Rightarrow \quad \frac{d\bar{J}_0}{ds} = -\frac{s\,\bar{J}_0}{s^2+1}$$

By separation of variables

$$\bar{J}_o = \frac{c}{\sqrt{s^2+1}}$$

Note that $\lim\limits_{s\to\infty} s\,\bar{f}(s) = f(0)$ **(Initial Value Theorem)**

$$\lim_{s\to\infty} s\,\bar{J}_o = J_0(0) = 1$$

we have

$$s\frac{c}{\sqrt{s^2+1}}\bigg|_{s=\infty} = 1 \qquad \Rightarrow \qquad c = 1$$

$$\therefore \quad \bar{J}_o = \mathcal{L}\{ J_0(t) \} = \frac{1}{\sqrt{s^2+1}}$$

[Exercise] Find $\mathcal{L}\{ t\, J_0(bt) \} = ??$

[Exercise] Find $\mathcal{L}\{ J_1(t) \}$ if $J_0'(t) = -J_1(t)$

[Exercise] Find $\mathcal{L}\{ e^{-at} J_0(bt) \}$

[Exercise] Find $\mathcal{L}\left\{ \dfrac{1 - J_0(t)}{t} \right\}$ Hint: $\displaystyle\int \frac{1}{\sqrt{s^2+1}}\, ds = \ln\left(s + \sqrt{s^2+1} \right)$

[Exercise] Find $\displaystyle\int_0^\infty J_0(t)\, dt$

[Exercise] Find $\mathcal{L}\{ t\, e^{-2t} J_1(t) \}$

[Exercise] Find $\displaystyle\int_0^\infty e^{-t} \left\{ \frac{1 - J_0(t)}{t} \right\} dt$

SUMMARY

0 $\mathcal{L}\{1\} = \dfrac{1}{s}$; $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$ for $n \in N$

$\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$; $\mathcal{L}\{\sin \omega t\} = \dfrac{\omega}{s^2+\omega^2}$; $\mathcal{L}\{\cos \omega t\} = \dfrac{s}{s^2+\omega^2}$

1 $\mathcal{L}\{a\,f(t) + b\,g(t)\} = a\,\mathcal{L}\{f(t)\} + b\,\mathcal{L}\{g(t)\}$

1′ $\mathcal{L}^{-1}\{a\,\bar{f}(s) + b\,\bar{g}(s)\} = a\,\mathcal{L}^{-1}\{\bar{f}(s)\} + b\,\mathcal{L}^{-1}\{\bar{g}(s)\} = a\,f(t) + b\,g(t)$

2 $\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0^+)$

Note that f(t) is continuous for $t \geq 0$ and f'(t) is piecewise continuous.

2′ If $f(0) = f'(0) = f''(0) = \ldots = f^{(n-1)}(0) = 0$, then

$\mathcal{L}^{-1}\{s^n\,\bar{f}(s)\} = f^{(n)}(t)$

3 $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \dfrac{1}{s}\,\mathcal{L}\{f(t)\} = \dfrac{\bar{f}(s)}{s}$

Question: what if the integration starts from a instead of 0?

3′ $\mathcal{L}^{-1}\left\{\dfrac{\bar{f}(s)}{s^n}\right\} = \int_0^t \ldots \int_0^t f(t)\,dt \ldots dt$

4 $\mathcal{L}\{t\,f(t)\} = -\bar{f}\,'(s)$; $\mathcal{L}\{t^n\,f(t)\} = (-1)^n\,\bar{f}^{(n)}(s)$

4′ $\mathcal{L}^{-1}\left\{ \frac{d^n}{ds^n} \bar{f}(s) \right\} = (-1)^n t^n f(t)$

5 $\mathcal{L}\left\{ \frac{f(t)}{t} \right\} = \int_s^\infty \bar{f}(\tilde{s})\, d\tilde{s}$ if $\frac{f(t)}{t}$ exists for $t \to 0$.

5′ $\mathcal{L}^{-1}\left\{ \int_s^\infty \bar{f}(\tilde{s})\, d\tilde{s} \right\} = \frac{f(t)}{t}$

6. $\mathcal{L}\{ e^{at} f(t) \} = \bar{f}(s-a)$

6′ $\mathcal{L}^{-1}\{ \bar{f}(s-a) \} = e^{at} f(t)$

7. $\mathcal{L}\{ f(t-a)\, u(t-a) \} = e^{-as} \bar{f}(s)$

7′ $\mathcal{L}^{-1}\{ e^{-as} \bar{f}(s) \} = f(t-a)\, u(t-a)$

8. $\mathcal{L}\{ u(t-a) \} = \frac{e^{-as}}{s}$; $\mathcal{L}\{ \delta(t-a) \} = e^{-as}$;

$\mathcal{L}\{ f \} = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\, dt$ where f(t) is a periodic function with period p

9. $\mathcal{L}\{ f(at) \} = \frac{1}{a} \bar{f}\left(\frac{s}{a}\right)$

9′ $\mathcal{L}^{-1}\{ \bar{f}(as) \} = \frac{1}{a} f\left(\frac{t}{a}\right)$

10. $\mathcal{L}\{ (f*g)(t) \} = \bar{f}(s)\, \bar{g}(s)$

10′ $\mathcal{L}^{-1}\{ \bar{f}(s)\, \bar{g}(s) \} = f*g$

where $(f*g)(t) \equiv \int_0^t f(t-\tau)\, g(\tau)\, d\tau$

11. $\lim_{t \to 0} f(t) = \lim_{s \to \infty} s\,\bar{f}(s)$; $\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\,\bar{f}(s)$