# A UNIFIED APPROACH FOR SOLVING NONLINEAR DIFFERENTIAL EQUATIONS IN PROCESS MODELS

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New insights in solving differential equations by the methods of orthogonal collocation and differential quadrature are presented here. These two methods are shown to be equivalent, if the trial functions are restricted to linearly independent polynomials of degree $(n-1)$ or less and the grid points are placed at the same locations. Explicit formulas of the quadrature coefficients are derived for arbitrarily-distributed grid points. A new grid point placement scheme is developed to simplify the quadrature coefficient formulas and to improve the accuracy of the numerical solutions. An example is provided to demonstrate the capability of the proposed techniques.

## INTRODUCTION

Nonlinear ordinary and partial differential equations are the most commonly used formulations in modelling process engineering systems. Except for very few cases, analytical solutions of nonlinear equations can not be easily obtained. Although various effective numerical techniques are available in the literature, one problem in implementing these methods is that each one of them is suitable for only a small class of equations and, thus, tends to be problem specific. Therefore, there is a need for developing a reliable, efficient, and also generic procedure that can be applied to a wide variety of problems.

Two existing techniques, i. e. the methods of orthogonal collocation and differential quadrature, seem to possess these desired properties. Orthogonal collocation has been widely implemented in solving chemical engineering problems, e. g. Villadsen and Stewart (1) and Finlayson (2). Excellent results have also been reported in several application studies of the method of differential quadrature, e. g. Mingle (3) and Bellman and Adomian (4). However, there are still a number of undesirable characteristics that are common to both approaches, e. g. the present method for calculating the quadrature coefficients becomes unstabe when the number of grid points is large and the solutions of differential equations near the boundaries tend to be less accurate due to the approximations of the boundary conditions.

Several new developments are presented in this paper. First, these two methods are shown to be equivalent, if the grid points are placed at the same locations. Second, explicit formulas for the differential quadrature coefficients at arbitrarily-located grid points are derived. Thus, the past problem of inverting an ill-conditioned matrix for determining the quadrature coefficients can be avoided. Also, the simple relation between the first-

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and second-order coefficients can be verified directly from these formulas. Third, a new strategy is developed for placing all the grid points, including the boundary points, at the zeros of orthogonal polynomials in an expanded interval. As a result, the explicit formulas can be further simplified and the corresponding calculation procedure becomes more efficient. Finally, the results of one numerical experiment are presented. In this example, the suggested grid point placement scheme appears to be superior to any other available choice, including the one adopted in the orthogonal collocation method.

## APPROXIMATION OF THE SPATIAL DERIVATIVES

The independent variables in a chemical engineering model are usually spatial distances ($y$) and/or time ($t$). The approaches of both the methods of orthogonal collocation and differential quadrature are the same, i. e. to simplify a given equation by approximating the spatial derivatives in terms of linear combinations of the solution at all grid points. More specifically,

$$\frac{\partial u(t,y_i)}{\partial y} \cong \sum_{j=1}^{n} \alpha_{ij} u(t,y_j) \qquad \frac{\partial^2 u(t,y_i)}{\partial y^2} \cong \sum_{j=1}^{n} \beta_{ij} u(t,y_j) \qquad i = 1,2,\ldots,n \tag{1}$$

where $u$ represents the dependent variable, $y_i$'s represent the locations of the grid points, $n$ is the number of grid points and the constants $\alpha_{ij}$'s and $\beta_{ij}$'s will be referred as first- and second-order quadrature coefficients in this paper. The values of the quadrature coefficients can be determined by assuming that, at a given value of $t$ and over an interval in $y$, a set of trial functions, $f_i(y)$ and $i = 1,2,\ldots,n$, can be chosen such that

$$u(t,y) \cong \sum_{i=1}^{n} \vartheta_i(t) f_i(y) \qquad i = 1,2,\ldots,n \tag{2}$$

where $\vartheta_i$, $i = 1,2,\ldots,n$, are constants to be determined. From Eqs. (1) and (2), it was shown in (2) and (4) that

$$\mathbf{A}^T = \mathbf{F}^{-1}\mathbf{S} \qquad \mathbf{B}^T = \mathbf{F}^{-1}\mathbf{D} \tag{3}$$

where,

$$\mathbf{A} = (\alpha_{ij})_{n\times n} \;\; \mathbf{B} = (\beta_{ij})_{n\times n} \;\; \mathbf{F} = \Big(f_i(y_j)\Big)_{n\times n} \quad \mathbf{S} = \left(\frac{d}{dy} f_i(y_j)\right)_{n\times n} \quad \mathbf{D} = \left(\frac{d^2}{dy^2} f_i(y_j)\right)_{n\times n}$$

where the $y_j$'s are the locations of grid points. In this paper, the entries in a matrix are always labeled by a pair of indices, $(i,j)$, referring to the $i$-th row and $j$-th column respectively. If the trial functions, $f_i(y) = y^{i-1}$ and $i = 1,2,\ldots,n$, are used, then the matrix $\mathbf{F}$ becomes the Vandermonde matrix $\mathbf{V}$ and $\mathbf{V} = (y_j^{i-1})_{n\times n}$. The corresponding matrices $\mathbf{S}$ and $\mathbf{D}$ become

$$\bar{\mathbf{V}} = \Big((i-1)y_j^{i-2}\Big)_{n\times n} \qquad \bar{\bar{\mathbf{V}}} = \Big((i-1)(i-2)y_j^{i-3}\Big)_{n\times n} \tag{4}$$

On the other hand, if the trial functions are arbitrary linearly independent polynomials, then the matrices $\mathbf{F}$, $\mathbf{S}$ and $\mathbf{D}$ can be written as

$$\mathbf{F} = \mathbf{C}\mathbf{V} \qquad \mathbf{S} = \mathbf{C}\bar{\mathbf{V}} \qquad \mathbf{D} = \mathbf{C}\bar{\bar{\mathbf{V}}} \tag{5}$$

where, $\mathbf{C} = (c_{ij})_{n\times n}$and $c_{ij}$ is the $j$-th coefficient of the $i$-th trial function. Substituting Eqs. (5) into Eqs. (3) yields

$$\mathbf{A}^T = \mathbf{V}^{-1}\bar{\mathbf{V}} \qquad \mathbf{B}^T = \mathbf{V}^{-1}\bar{\bar{\mathbf{V}}} \tag{6}$$

Note that the matrices **A** and **B** in Eqs. (6) are the same as those obtained by using the trial functions $y^{i-1}$ and $i = 1, 2, \ldots n$. This result implies that, as long as linearly independent polynomials of degree $(n-1)$ or less are used as trial functions, the values of the quadrature coefficients are dependent only on the locations of the grid points. Thus, the methods of collocation and the differential quadrature are essentially equivalent if the same set of grid points is adopted.

Also, notice that the Vandermonde matrix is known to be ill-conditioned for large value of $n$. It has been well-documented that significant errors can be introduced if the quadrature coefficients are determined by using Eqs. (6), e. g. (3) and Civan and Sliepcevich (5). Thus, there is a definite need for developing an alternative procedure to calculate the quadrature coefficients.

## THE GENERAL FORMULAS

Let's consider a special form of trial functions:

$$\tilde{f}_i(y) = \frac{\phi_n(y)}{(y-y_i)(d\phi_n(y_i)/dy)} = \sum_{j=1}^{n} \tilde{c}_{ij} y^{j-1} \qquad i = 1, 2, \ldots, n \tag{7}$$

where

$$\phi_n(y) = \prod_{m=1}^{n} (y - y_m) \tag{8}$$

Note, if the grid points are chosen at the zeros of $\phi_n(y)$, the coefficient matrix of these trial functions is exactly the inverse of Vandermonde matrix (Hamming (6)), i. e. $\mathbf{V}^{-1} = \tilde{\mathbf{C}} = (\tilde{c}_{ij})_{n\times n}$. Thus, Eq. (6) can be changed to

$$\mathbf{A}^T = \tilde{\mathbf{C}}\bar{\mathbf{V}} = \tilde{\mathbf{S}} \qquad \mathbf{B}^T = \tilde{\mathbf{C}}\bar{\bar{\mathbf{V}}} = \tilde{\mathbf{D}} \tag{9}$$

where,

$$\tilde{\mathbf{S}} = \left(\frac{d}{dy}\tilde{f}_i(y_j)\right)_{n\times n} \qquad \tilde{\mathbf{D}} = \left(\frac{d^2}{dy^2}\tilde{f}_i(y_j)\right)_{n\times n}$$

From Eqs. (7)-(9), it can be concluded that the values of the first, second and third derivatives of the function $\phi_n$ at the grid points need to be calculated in order to determine the quadrature coefficients. A recursive procedure was suggested by Villadsen and Michelsen (7) for this purpose. However, a more efficient and more accurate approach can be obtained if the analytical expressions of these derivatives are available. They were derived by directly differntiating Eq. (8) in Quan (8).

If these explicit expressions are substituted into Eqs. (9), then

$$\alpha_{ij} = \frac{1}{y_j - y_i} \prod_{\substack{m=1 \\ m\neq i,j}}^{n} \frac{y_i - y_m}{y_j - y_m} \qquad \beta_{ij} = \frac{2}{y_j - y_i}\left(\prod_{\substack{m=1 \\ m\neq i,j}}^{n} \frac{y_i - y_m}{y_j - y_m}\right)\left(\sum_{\substack{k=1 \\ k\neq i,j}}^{n} \frac{1}{y_i - y_k}\right) \tag{10}$$

for $i \neq j$, and

$$\alpha_{ii} = \sum_{\substack{k=1 \\ k\neq i}}^{n} \frac{1}{y_i - y_k} \qquad \beta_{ii} = 2\sum_{\substack{k=1 \\ k\neq i}}^{n-1}\left[\frac{1}{y_i - y_k}\left(\sum_{\substack{l=k+1 \\ l\neq i}}^{n} \frac{1}{y_i - y_l}\right)\right] \tag{11}$$

for $i = j$.

As far as the authors are aware, the results presented in Eqs. (10) and (11) are new. They will be referred as the general formulas. These formulas are valid for arbitrary location of grid points. More significantly, the differential quadrature coefficients can now be calculated directly and, therefore, accurate solutions can be obtained even for large values of $n$.

One of the benefits of having these explixit formulas is that several important properties of this approach can be easily verified. For example, it can be shown from Eqs. (10) and (11) that

$$\mathbf{B} = \mathbf{A}^2 \tag{12}$$

if the trial functions are $n$ linearly independent polynomials of degree $(n-1)$ or less. A rigorous proof of this equation, which has not been published before, is presented in the Appendix. Previous justification of this relation (4) is intuitive and, furthermore, implies that it is valid for all types of trial functions.

## GRID POINT PLACEMENT SCHEME

As mentioned before, the values of the quadrature coefficients are dependent upon the locations of the nodes. It should be pointed out that the distributions of these grid points also affect the accuracy of the numerical solutions. Other than evenly-placed nodes, the zeros of orthogonal polynomials (shifted Legendre) were adopted as the grid points in several previous studies of the differential quadrature method, e. g. (4) and Bellman and Roth (9). In these cases, the interval of the orthogonal polymomial, $[a, b]$, coincides with the actual spatial interval of the differential equation, $[C_1, C_2]$. An example of this grid point placement scheme is presented in Figure 1(i), in which the roots of a fifth-degree shifted Legendre polynomial are used in the interval $[0, 1]$. Since the boundary points are not included as grid points, additional approximations need to be introduced in imposing the given boundary conditions. As a result, the accuracy of the published results near boundaries is not satisfactory.

The method of orthogonal collocation (2) selects, as grid points, the zeros of an orthogonal polynomial of degree $(n-2)$ within the interval $[C_1, C_2]$. Again, the actual and the polynomial intervals are the same. Two additional grid points are then placed at $C_1$ and $C_2$ to satisfy the boundary conditions. This approach is illustrated in Figure 1(ii), in which the roots of a third-degree shifted Legendre polynomial are used as the interior nodes. In this case, although the boundary conditions are handled more appropriately, the overall accuracy of the solutions can still be improved , e. g. see Burka (10).

Note that, Eq. (2) can be regarded as the approximation of the dependent variable $u$ by Lagrange interpolation, if Eqs. (7) are used as the trial functions and if

$$\vartheta_i(t) = u(t, y_i) \qquad i = 1, 2, \ldots, n \tag{13}$$

It has been well-established that the absolute value of the error of interpolation is nearly minimized in the minimax sense, if the interpolation points are chosen to be the zeros of a Chebyshev polynomial of the first kind, e. g. see Phillips and Taylor (11). Thus, although the accuracy of using the approximations suggested by Eq. (1) is still not guaranteed, the zeros of the Chebyshev polynomials (first kind) seem to be the most reasonable choice for grid points.

To avoid the drawbacks of the methods of differential quadrature and orthogonal collocation, a new strategy has been developed for placing the grid points. Since the zeros of the orthogonal polynomials are located within an interval $[a, b]$, it is best to choose $a$ and

$b$ such that the first and last roots coincide with the lower and upper bounds of the actual interval $[C_1, C_2]$. To be more specific,

$$a = \frac{2[C_2(1+x_1) - C_1(1+x_n)]}{(1-x_n)(1+x_1) - (1-x_1)(1+x_n)} \qquad b = \frac{2[C_1(1-x_n) - C_2(1-x_1)]}{(1-x_n)(1+x_1) - (1-x_1)(1+x_n)} \quad (14)$$

where $x_1$ and $x_n$ are the first and last zeros of an unshifted orthogonal polynomial defined in the interval $[-1, 1]$. The shifted roots,

$$y_i = \left(\frac{b-a}{2}\right)x_i + \left(\frac{b+a}{2}\right) \qquad i = 1, 2, \ldots, n \quad (15)$$

can then be chosen as the grid points. An example is presented in Figure 1(iii), in which the zeros of a fifth-degree Chebyshev polynomial are used as grid points. Note that the polynomial interval $[-0.026, 1.026]$ is different from the actual interval $[0, 1]$ in this case. It has been shown in numerous numerical experiments (8) that placement of grid points in this fashion results in more accurate solutions than any other method. One example is provided in the next section to demonstrate this phenomenon.

One of the additional advantages of using the proposed scheme is that the general formulas for determining the quadrature coefficients can be further simplified. For example, if the roots of a $n$-th degree Chebyshev (first kind) polynomials are adopted as the grid points, the simplified formulas can be written as

$$\alpha_{ij} = (-1)^{i-j}\frac{2}{(b-a)(x_i - x_j)}\sqrt{\frac{1-x_j^2}{1-x_i^2}} \quad (16a)$$

$$\beta_{ij} = (-1)^{i-j}\frac{4[x_i(x_i - x_j) - 2(1-x_i^2)]\sqrt{(1-x_i^2)(1-x_j^2)}}{(b-a)^2(x_i - x_j)^2(1-x_i^2)^2} \quad (16b)$$

for $i \neq j$ and

$$\alpha_{ii} = \frac{x_i}{(b-a)(1-x_i^2)} \qquad \beta_{ii} = \frac{4[(2+n^2)x_i^2 + 1 - n^2]}{3(b-a)^2(1-x_i^2)^2} \quad (17)$$

for $i = j$, where $x_i$ and $x_j$ represent the unshifted zeros of a $n$-th degree Chebyshev polynomial of the first kind defined within $[-1, 1]$. Their values can be calculated easily by

$$x_{n-i+1} = \cos\left(\frac{2i-1}{2}\frac{\pi}{n}\right) \qquad i = 1, 2, \ldots, n \quad (18)$$

The derivations of the quadrature coefficient formulas corresponding to the Legendre, Chebyshev of the first and second kind, Ultraspherical and Jacobi polynomials are published elsewhere by Quan and Chang (12). It should be emphasized that these simplifications can not be realized if the placement scheme of the orthogonal collocation is used. The formulas presented in Eqs. (16a) and (16b) are new and should be considered to be improved versions of the published results in (12).

## SOLUTIONS OF DIFFERENTIAL EQUATIONS

After approximating the spatial derivatives, one can transform the equations under consideration to a simpler problem. Let's consider a general form of differential equations:

$$\frac{\partial u}{\partial t} = \psi\left(t, u, y, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial y^2}, z, \frac{\partial u}{\partial z}, \frac{\partial^2 u}{\partial z^2}, \ldots, etc.\right) \quad (19)$$

with appropriate initial and boundary conditions. If all the spatial derivatives are approximated by Eqs. (1), Eq. (19) can be written as a set of ordinary differential equations. Similarly, the boundary conditions can be approximated by algebraic equations for a given $t$ by using Eqs. (1). Thus, these resulting equations form an initial-value problem and can be integrated by a standard algorithm, such as the Runge-Kutta procedure. For "steady state" problems, the time derivative of $u$ on the left hand side equals zero. A set of algebraic equations can be obtained by approximating all the spatial derivatives. Standard numerical methods, such as Newton-Raphson, can be used to solve this problem. Alternatively, the steady state problems can be solved by applying Eqs. (1) to all but one spatial derivatives. As a result, a two-point boundary-value problem can be formulated (8). Again, standard procedures, e. g. the shooting method, are readily available. The following example is presented to demonstrate the improvement in accuracy achieved by using the proposed method.

## Example

The simplest form of Eq. (19) is a steady state problem with spatial derivatives only in one direction, i. e. boundary-value problems formulated by ordinary differential equations. One such problem was reported in Finlayson (2):

$$\frac{d}{dy}\left[(1+u)\frac{du}{dy}\right] = 0 \qquad (20)$$

The boundary conditions are:

$$u(0) = 0 \qquad u(1) = 1 \qquad (21)$$

This problem was solved by using three grid point placement schemes, i. e. equally-spaced intervals (method 1), orthogonal collocation (method 2) and the proposed apporach (method 3). In each case, five grid points were adopted. A summary of the solutions at the three interior points are presented in Table 1. The values of the relative errors, $\epsilon$, were determined by

$$\epsilon = \frac{|u^c - u^a|}{|u^a|} \qquad (22)$$

where, $u^c$ and $u^a$ represent the numerical and the analytical solutions respectively. The arithemetic means of the relative errors in these three cases are 0.00145, 0.00275 and 0.00058 respectively. It can be clearly seen from these data that, when compared with the analytical solutions, the results obtained by using the proposed method are more accurate than the others.

**TABLE 1 -Summary of Solutions**

| Method | Grid Point Locations | Numerical Solution | Analytical Solution | $\epsilon$ |
|---|---|---|---|---|
| | 0.250000 | 0.321754 | 0.322876 | 0.003473 |
| 1 | 0.500000 | 0.580885 | 0.581139 | 0.000437 |
| | 0.750000 | 0.803127 | 0.802776 | 0.000437 |
| | 0.112702 | 0.157867 | 0.156765 | 0.007030 |
| 2 | 0.500000 | 0.581579 | 0.581139 | 0.000758 |
| | 0.887298 | 0.913195 | 0.913608 | 0.000452 |
| | 0.190983 | 0.253777 | 0.254173 | 0.001558 |
| 3 | 0.500000 | 0.581097 | 0.581139 | 0.000072 |
| | 0.809017 | 0.851323 | 0.851230 | 0.000110 |

## CONCLUSIONS

New insights in the methods of orthogonal collocation and differential quadrature are presented in this paper. It is shown that the quadrature coefficients are dependent only upon the locations of the grid points as long as linearly independent polynomials of degree $(n-1)$ or less are adopted as trial functions. The general formulas are useful not only for determining quadrature coefficients accurately, but also for verifying properties which are important to both methods. The advantages of using the proposed grid point placement scheme are demonstrated in a numerical example. It should be pointed out that numerous other examples have been carried out (8) and the same conclusions can be drawn from the results of these studies. Thus, it appears that the proposed approach is better than any other available choices in terms of accuracy and efficiency.

## SYMBOLS

English Letters

| | |
|---|---|
| **A** | = the first-order quadrature coefficient matrix |
| **B** | = the second-order quadrature coefficient matrix |
| $a,b$ | = the lower and upper bound of a finite interval |
| $C_1,C_2$ | = the lower and upper bound of the actual interval of a differential equations |
| $f_i$ | = the $i$-th trial function |
| $n$ | = total number of grid points |
| $T_n$ | = Chebyshev polynomial of the first kind of degree n |
| $t$ | = independent variable, $0 \leq t < \infty$ |
| $u$ | = dependent variable |
| **V** | = the Vandermonde matrix |
| $x$ | = independent variable, distance in rectangular coordinates $-1 \leq x \leq 1$ |
| $y$, $z$ | = independent variable, spatial distances in rectangular coordinates |

Greek Letters

| | |
|---|---|
| $\alpha_{ij}$ | = the first-order quadrature coefficients |
| $\beta_{ij}$ | = the second-order quadrature coefficients |
| $\epsilon$ | = the relative error of numerical solution |
| $\vartheta_i$ | = coefficient of the $i$-th trial function |
| $\phi_n$ | = a polynomial with $n$ real roots |

## REFERENCES

1. Villadsen, J. V. , and Stewart, W. E. , 1967, Chem. Eng. Sci. 22, 1483.

2. Finlayson, B. A. , 1980, "Nonlinear Analysis in Chemical Engineering", McGraw-Hill, New York.

3. Mingle, J. O. , 1977, J. Math. Anal. Appl. 60, 559.

4. Bellman, R. E. , and Adomian, G. , 1985, "Partial Differential Equations", D. Reidel Publishing Company, Holland.

5. Civan, F. , and Sliepcevich, C. M. , 1983, J. Math. Anal. Appl. 93, 206.

6. Hamming, R. W. , 1977, "Numerical Methods for Scientists and Engineers", 2nd ed. , McGraw-Hill, New York.

7. Villadsen, J. V. , and Michelsen, M. L. , 1978, "Solution of Differential Equation Models by Polynomial Approximation", Prentice-Hall, New Jersey.

8. Quan, J. R. , 1988, M. S. Thesis, University of Nebraska - Lincoln.

9. Bellman, R. E. , and Roth, R. S. , 1986, "Methods in Approximation", D. Reidel Publishing Company, Holland.

10. Burka, M. A. , 1982, AIChE J. 28, 11.

11. Philips, G. M. and Taylor, P. J. , 1973, "Theory and Applications of Numerical Analysis", Academic Press, London.

12. Quan, J. R. and Chang, C. T. , 1989, Comput. & Chem. Engng. (in press).

# APPENDIX

The proof of the relation, $\mathbf{B} = \mathbf{A}^2$, consists of two parts, i. e.

$$\beta_{ij} = \sum_{k=1}^{n} \alpha_{ik}\alpha_{kj} \qquad i \neq j \tag{A1}$$

$$\beta_{ii} = \sum_{k=1}^{n} \alpha_{ik}\alpha_{ki} \qquad i = j \tag{A2}$$

First, the right-hand side of Eq. (A1) can be written as

$$\sum_{k=1}^{n} \alpha_{ik}\alpha_{kj} = \sum_{\substack{k=1\\k\neq i,j}}^{n} \alpha_{ik}\alpha_{kj} + (\alpha_{ii} + \alpha_{jj})\alpha_{ij} \tag{A3}$$

Substituting the first equation in Eqs. (10) into the first term on the right hand side of Eq. (A3) yields

$$\sum_{\substack{k=1\\k\neq i,j}}^{n} \alpha_{ik}\alpha_{kj} = \sum_{\substack{k=1\\k\neq i,j}}^{n} \left[\left(\frac{1}{y_k - y_i} \prod_{\substack{m=1\\m\neq i,k}}^{n} \frac{y_i - y_m}{y_k - y_m}\right)\left(\frac{1}{y_j - y_k} \prod_{\substack{m=1\\m\neq k,j}}^{n} \frac{y_k - y_m}{y_j - y_m}\right)\right]$$

$$= \left(\prod_{\substack{m=1\\m\neq i,j}}^{n} \frac{y_i - y_m}{y_j - y_m}\right)\left(\sum_{\substack{k=1\\k\neq i,j}}^{n} \frac{1}{(y_i - y_k)(y_j - y_k)}\right)$$

$$= \frac{1}{y_j - y_i}\left(\prod_{\substack{m=1\\m\neq i,j}}^{n} \frac{y_i - y_m}{y_j - y_m}\right)\left(\sum_{\substack{k=1\\k\neq i,j}}^{n} \frac{y_j - y_i}{(y_i - y_k)(y_j - y_k)}\right) \tag{A4}$$

Substituting the first equations in Eqs. (10) and (11) into the second term on the right-hand side of Eq. (A3) gives

$$\alpha_{ij}(\alpha_{ii}+\alpha_{jj}) = \frac{1}{y_j - y_i}\left(\prod_{\substack{m=1\\m\neq i,j}}^{n} \frac{y_i - y_m}{y_j - y_m}\right)\left(\sum_{\substack{k=1\\k\neq i}}^{n}\frac{1}{y_i - y_k} + \sum_{\substack{k=1\\k\neq j}}^{n}\frac{1}{y_j - y_k}\right)$$

$$= \frac{1}{y_j - y_i}\left(\prod_{\substack{m=1\\m\neq i,j}}^{n} \frac{y_i - y_m}{y_j - y_m}\right)\left[\sum_{\substack{k=1\\k\neq i,j}}^{n}\left(\frac{1}{y_i - y_k} + \frac{1}{y_j - y_k}\right)\right] \tag{A5}$$

Combinning Eqs. (A4) and (A5), one can obtain

$$\sum_{k=1}^{n}\alpha_{ik}\alpha_{kj} = \frac{1}{y_j - y_i}\left(\prod_{\substack{m=1\\m\neq i,j}}^{n} \frac{y_i - y_m}{y_j - y_m}\right)\left[\sum_{\substack{k=1\\k\neq i,j}}^{n}\frac{y_j - y_i}{(y_i - y_k)(y_j - y_k)} + \sum_{\substack{k=1\\k\neq i,j}}^{n}\left(\frac{1}{y_i - y_k} + \frac{1}{y_j - y_k}\right)\right]$$

$$= \frac{1}{y_j - y_i}\left(\prod_{\substack{m=1\\m\neq i,j}}^{n} \frac{y_i - y_m}{y_j - y_m}\right)\left(\sum_{\substack{k=1\\k\neq i,j}}^{n}\frac{2}{y_i - y_k}\right) = \beta_{ij} \tag{A6}$$

This completes the first part of the proof. Similarly, the right-hand side of Eq. (A2) can be written as

$$\sum_{k=1}^{n}\alpha_{ik}\alpha_{ki} = \alpha_{ii}^2 + \sum_{\substack{k=1\\k\neq i}}^{n}\alpha_{ik}\alpha_{ki} \tag{A7}$$

By substituting the first equations in Eqs. (10) and (11), Eq. (A7) can be changed to

$$\sum_{k=1}^{n}\alpha_{ik}\alpha_{ki} = \left(\sum_{\substack{k=1\\k\neq i}}^{n}\frac{1}{y_i - y_k}\right)^2 - \sum_{\substack{k=1\\k\neq i}}^{n}\frac{1}{(y_i - y_k)^2}$$

$$= \sum_{\substack{k=1\\k\neq i}}^{n}\left[\frac{1}{y_i - y_k}\left(\sum_{\substack{l=1\\l\neq i}}^{n}\frac{1}{y_i - y_l}\right) - \frac{1}{(y_i - y_k)^2}\right]$$

$$= \sum_{\substack{k=1\\k\neq i}}^{n}\left[\frac{1}{y_i - y_k}\left(\sum_{\substack{l=1\\l\neq i,k}}^{n}\frac{1}{y_i - y_l}\right)\right] \tag{A8}$$

Since all combinations of $l$'s and $k$'s in Eq. (A8) are repeated twice, an equivalent form of Eq. (A9) can be written

$$\sum_{k=1}^{n}\alpha_{ik}\alpha_{ki} = 2\sum_{\substack{k=1\\k\neq i}}^{n-1}\left[\frac{1}{y_i - y_k}\left(\sum_{\substack{l=k+1\\l\neq i}}^{n}\frac{1}{y_i - y_l}\right)\right] = \beta_{ii} \tag{A9}$$

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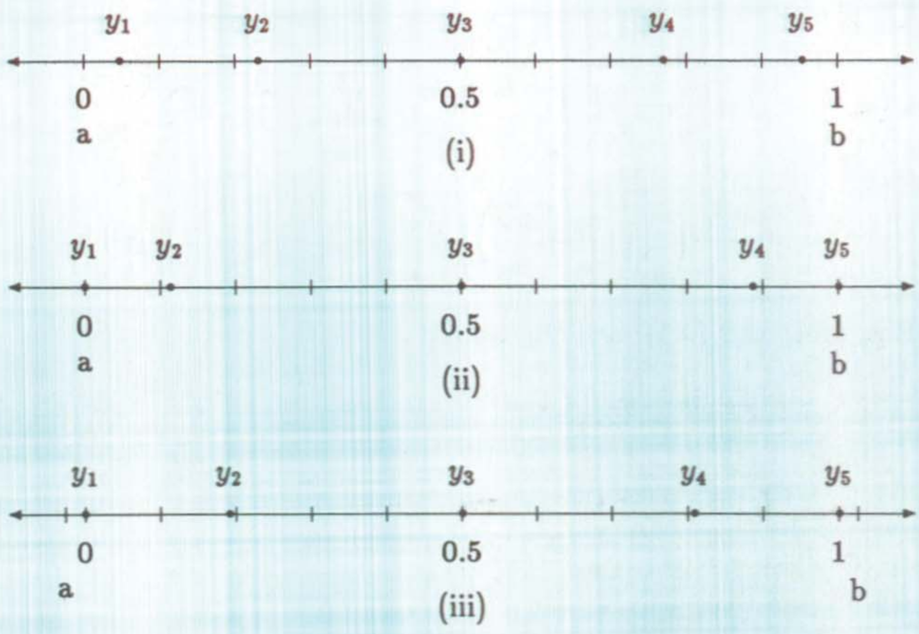


**Figure 1. Location of grid points: (i) differential quadrature ( Legendre ); (ii) orthogonal collocation ( Legendre ); (iii) suggested approach ( Chebyshev ).**