Game Theory

Chapter 3
Two-Person Nonzero Sum Games

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The Basics

- We no longer assume that the game is zero sum, or even constant sum. All players will have their own individual payoff matrix and the goal of maximizing their own individual payoff.

    - Suppose that the payoff matrices are

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix}.$$

    - In a zero sum game we always had $a_{ij} + b_{ij} = 0,$ or $a_{ij} + b_{ij} = k$ where k is a fixed constant, but now in a nonzero sum game we do not assume that.

The Basics

- The payoff when player I plays row *i* and player II plays column *j* is now a pair of numbers (a_{ij}, b_{ij}), where the first component is the payoff to player I and the second number is the payoff to player II.

- The individual rows and columns are called pure strategies for the players.

- Every zero sum game can be put into the bimatrix framework by taking B=-A, so this is true generalization of the theory in the first chapter.

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Example 3.1

- Two students have an exam tomorrow. They can choose to study, or go to a party. The payoff matrices, written together as a bimatrix, are given by

I/II	Study	Party
Study	$(2, 2)$	$(3, 1)$
Party	$(1, 3)$	$(4, -1)$

  - A mixed strategy for player I is $X = (x_1, \ldots, x_n) \in S_n$ with $x_i \geq 0$, the probability that player I uses row i, and so $x_1 + x_2 + \cdots + x_n = 1$. Similarly for player II, $Y = (y_1, \ldots, y_m) \in S_m$, with $y_j \geq 0$ and $y_1 + \cdots + y_m = 1$.

  - Expected payoffs

$$E_I(X, Y) = XAY^T \text{ for player I,}$$
$$E_{II}(X, Y) = XBY^T \text{ for player II.}$$

Nash Equilibrium

- **Definition 3.1.1** *A pair of mixed strategies* $(X^* \in S_n, Y^* \in S_m)$ *is a Nash equilibrium if* $E_I(X, Y^*) \leq E_I(X^*, Y^*)$ *for every mixed* $X \in S_n$ *and* $E_{II}(X^*, Y) \leq E_2(X^*, Y^*)$ *for every mixed* $Y \in S_m$. *If* (X^*, Y^*) *is a Nash equilibrium we denote by* $v_A = E_I(X^*, Y^*)$ *and* $v_B = E_{II}(X^*, Y^*)$ *as the optimal payoff to each player. Written out with the matrices,* (X^*, Y^*) *is a Nash equilibrium if*

$$E_I(X^*, Y^*) = X^* A Y^{*T} \geq X A Y^{*T} = E_I(X, Y^*), \text{ for every } X \in S_n,$$

$$E_{II}(X^*, Y^*) = X^* B Y^{*T} \geq X^* B Y^T = E_{II}(X^*, Y), \text{ for every } Y \in S_m.$$

  - Neither player can gain any expected payoff if either one chooses to deviate from playing the Nash equilibrium, **assuming that the other player is implementing his or her piece of the Nash equilibrium**.
  - Each strategy in a Nash equilibrium is a **best response strategy** against the opponent's Nash strategy.

Best Response Strategy

- **Definition 3.1.2** *A strategy $X^0 \in S_n$ is a* **best response strategy** *to a given strategy $Y^0 \in S_m$ for player II, if*

$$E_I(X^0, Y^0) = \max_{X \in S_n} E_I(X, Y^0).$$

Similarly, a strategy $Y^0 \in S_m$ is a **best response strategy** *to a given strategy $X^0 \in S_n$ for player I, if*

$$E_{II}(X^0, Y^0) = \max_{Y \in S_m} E_{II}(X^0, Y).$$

- – X^* is a best response to Y^* and Y^* is a best response to X^*.
- – If $B = -A$, a bimatrix game is a zero sum two-person game and a Nash equilibrium is the same as a saddle point in mixed strategies.
  - $E_I(X, Y) = XAY^T = -E_{II}(X, Y).$

A Nash Equilibrium in Pure Strategies

- In the bimatrix game a Nash equilibrium in pure strategies must be the pair that is, **the largest first component in the column and the largest second component in the row.**
  - A Nash equilibrium in pure strategies will be a row i^* and column j^* satisfying

    $$a_{ij^*} \leq a_{i^*j^*} \quad \text{and} \quad b_{i^*j} \leq b_{i^*j^*}, i = 1, \ldots, n, j = 1, \ldots, m.$$

  - $a_{i^*j^*}$ is the largest number in column j^* and $b_{i^*j^*}$ is the largest number in row i^*.

A Nash Equilibrium in Pure Strategies (cont'd)

- Strategies and payoffs
  - If player I uses the pure strategy row i, and player II uses the mixed strategy Y, then the expected payoffs to each player are

  $$E_I(i, Y) = {}_i AY^T \quad \text{and} \quad E_{II}(i, Y) = {}_i BY^T.$$

  - If player II uses the pure strategy column j, and player I uses the mixed strategy X, then the expected payoffs to each player are

  $$E_I(X, j) = XA_j \quad \text{and} \quad E_{II}(X, j) = XB_j.$$

- Questions for a given bimatrix game
  - Is there a Nash equilibrium using pure strategies?
  - Is there a Nash equilibrium using mixed strategies? More than one?
  - How to compute these?

Prisoner's Dilemma

- Two criminals have just been caught after committing a crime. The police interrogate the prisoners by placing them in separate rooms so that they cannot communicate and coordinate their stories. The goal of the police is to try to get one or both of them to confess to having committed the crime. We consider the two prisoners as the players in a game in which they have two pure strategies: confess, or don't confess. The following matrix represents the possible payoffs.

Prisoner I/II	Confess	Don't confess
Confess	$(-5, -5)$	$(0, -20)$
Don't confess	$(-20, 0)$	$(-1, -1)$

Prisoner's Dilemma (cont'd)

- The individual matrices for the two prisoners are

$$A = \begin{bmatrix} -5 & 0 \\ -20 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} -5 & -20 \\ 0 & -1 \end{bmatrix}.$$

- The numbers are negative because they represent the number of prison sentence and each player wants to maximize the payoff.

- **Systematic** way to find the payoff pair (a,b)
  - Put a bar over the first number that is the largest in each column and put a bar over the second number that is the largest in each row.
  - Any pair of numbers that <u>both have bars</u> is a Nash equilibrium in pure strategies.

Prisoner's Dilemma (cont'd)

Prisoner I/II	Confess	Don't confess
Confess	$(-\overline{5}, -\overline{5})$	$(\overline{0}, -20)$
Don't confess	$(-20, \overline{0})$	$(-1, -1)$

- There is exactly one pure Nash equilibrium at (confess, confess), where the payoff pair (-5, -5) is **stable** because neither player can improve their own individual payoff if they both play it.

- The players are rewarded for a betrayal of the other prisoner, and so that is exactly what will happen.
  - This reveals a major reason why conspiracies almost always fail.

- The payoff pair (-1, -1) is unstable in the sense that a player can do better by deviating, assuming that the other player does not.

Prisoner's Dilemma (cont'd)

- Whereas the payoff pair (-5,-5) is stable because neither player can improve their own individual payoff if they both play it.

- The Nash equilibrium is self-enforcing.
  - It would take extraordinary with power for both players to stick with that agreement in the face of the numbers.

- This problem can be solved by **domination**.
  - For player I, row 1 strictly dominates row 2.
  - For player II, column 1 strictly dominates column 2.

Example 3.2

- Go back to the study-party game and change one number:

I/II	Study	Party
Study	(2, 2)	(3, 1)
Party	(1, 3)	(4, 4)

- There are Nash equilibria at payoff (2,2) and at (4,4).
- A bimatrix game can have more than one Nash equilibrium!

Example 3.3

- **The Arms Race.** Suppose that two countries have the choice of <u>developing or not developing nuclear weapons</u>. There is a cost of the development of the weapons in the price that the country might have to pay in sanctions, and so forth. But there is also a benefit in having nuclear weapons in prestige, defense, deterrence, and so on. We quantify the game using a bimatrix in which each player wants to maximize the payoff.

Country I/ II	Nuclear	Conventional
Nuclear	$(1, 1)$	$(10, -5)$
Conventional	$(-5, 10)$	$(1, 1)$

Example 3.3 (cont'd)

- There is a Nash equilibrium at the pair (1,1) corresponding to the strategy (nuclear, nuclear).

- The pair (1,1) when both countries maintain conventional weapons is not a Nash equilibrium because each player can improve its own payoff by unilaterally deviating from this.

- Once one government obtains nuclear weapons, it is a Nash equilibrium-and self-enforcing equilibrium-for opposing countries to also obtain the weapons.

Example 3.4

- Consider the game with matrix

$$\begin{bmatrix} (2,0) & (1,3) \\ (0,1) & (3,0) \end{bmatrix}.$$

- There is <u>no pair (*a*, *b*)</u> in which *a* is the largest in the column and *b* is the largest in the row.
- Not all bimatrix games have Nash equilibrium in pure strategies!
- It seems reasonable that we use mixed strategies.
- Even though a game might have pure strategy Nash equilibria, it could also have a mixed strategy Nash equilibrium.

Safety Value

- **Definition 3.1.3** *Consider the bimatrix game with matrices (A, B). The* **safety value** *for player I is* $value(A)$. *The* **safety value** *for player II in the bimatrix game is* $value(B^T)$.

  If A has the saddle point (X^A, Y^A), then X^A is called the **maxmin strategy for player I**.

  If B^T has saddle point (X^{B^T}, Y^{B^T}), then X^{B^T} is the **maxmin strategy for player II.**.

  – The safety levels are the <u>guaranteed amounts each player can get</u> by using their own individual maxmin strategies, so any rational player must get at least the safety level in a bimatrix game.

Safety Value (cont'd)

- Example: In the game with matrix

$$\begin{bmatrix} (2,0) & (1,3) \\ (0,1) & (3,0) \end{bmatrix},$$

  we have
$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}, B^T = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}.$$

  - $v(A) = \frac{3}{2}$ is the safety value for player I and $v(B^T) = \frac{1}{4}$ is the safety value for player II.
  - The maxmin strategy for player I is $X = (\frac{3}{4}, \frac{1}{4})$, so if player I uses $X = (\frac{3}{4}, \frac{1}{4})$, then $E_{\mathrm{I}}(X, Y) \geq v(A) = \frac{3}{2}$ no matter what Y strategy is used.
    $$E_{\mathrm{I}}\left(\left(\frac{3}{4}, \frac{1}{4}\right), Y\right) = \frac{3}{2}(y_1 + y_2) = \frac{3}{2}, \text{ for any strategy } Y = (y_1, y_2).$$
  - The maxmin strategy for player II is $Y = X^{B^T} = (\frac{1}{4}, \frac{3}{4})$ with safety $\frac{1}{4}$.

Safety Value (cont'd)

- **Individually rational**

  It has to be true that if (X^*, Y^*) is a Nash equilibrium for the bimatrix game (A, B), then

  $$E_{\mathrm{I}}(X^*, Y^*) = X^* A Y^{*T} \geq value(A) \text{ and } E_{\mathrm{II}}(X^*, Y^*) = X^* B Y^{*T} \geq value(B^T).$$

  - In the bimatrix game, if players use their Nash points, they get at least their safety levels.

Safety Value (cont'd)

- **Proof.** It's really just from the definitions. The definition of Nash equilibrium says

$$E_{\mathrm{I}}(X^*, Y^*) = X^* A Y^{*T} \geq E_{\mathrm{I}}(X, Y^*) = X A Y^{*T}, \text{ for all } X \in S_n.$$

But if that is true for all mixed X, then

$$E_{\mathrm{I}}(X^*, Y^*) \geq \max_{X \in S_n} X A Y^{*T} \geq \min_{Y \in S_m} \max_{X \in S_n} X A Y^T = value(A).$$

The other part of a Nash definition gives us

$$E_{\mathrm{II}}(X^*, Y^*) = X^* B Y^{*T} \geq \max_{Y \in S_m} X^* B Y^T$$

$$= \max_{Y \in S_m} Y B^T X^{*T} \qquad (\text{since } X^* B Y^T = Y B^T X^{*T})$$

$$\geq \min_{X \in S_n} \max_{Y \in S_m} Y B^T X^T = value(B^T).$$

Each player does at least as well as assuming the worst. □

2×2 Bimatrix Games ◁

Two-Person 2 x 2 Nonzero Sum Games

- Mixed strategies
  - Let $X = (x, 1 - x), Y = (y, 1 - y), 0 \leq x \leq 1, 0 \leq y \leq 1$ be mixed strategies for players I and II, respectively.
  - Expected payoffs

$$E_I(X, Y) = XAY^T = (x, \; 1 - x) \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y \\ 1 - y \end{bmatrix}$$

$$E_{II}(X, Y) = XBY^T = (x, \; 1 - x) \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} y \\ 1 - y \end{bmatrix}$$

  - It is the goal of each player to maximize her own expected payoff **assuming that the other player is doing her best to maximize her own payoff** with the strategies she controls.

Conditions for a Nash Equilibrium Point

- **Proposition 3.2.1** *A necessary and sufficient condition for $X^* = (x^*, 1-x^*), Y^* = (y^*, 1-y^*)$ to be a Nash equilibrium point of the game with matrices (A, B) is*

$$
\begin{array}{llcl}
(1) & E_I(1, Y^*) & \leq & E_I(X^*, Y^*) \\
(2) & E_I(2, Y^*) & \leq & E_I(X^*, Y^*) \\
(3) & E_{II}(X^*, 1) & \leq & E_{II}(X^*, Y^*) \\
(4) & E_{II}(X^*, 2) & \leq & E_{II}(X^*, Y^*)
\end{array}
$$

  - To find the Nash equilibria we need to find all solutions of the inequalities (1)-(4).

Conditions for a Nash Equilibrium Point

Proof. To see why this is true, we first note that if (X^*, Y^*) is a Nash equilibrium, then the inequalities must hold by definition (simply choose pure strategies for comparison). So we need only show that the inequalities are sufficient.

Suppose that the inequalities hold for (X^*, Y^*). Let $X = (x, 1 - x)$ and $Y = (y, 1 - y)$ be any mixed strategies. Successively multiply (1) by $x \geq 0$ and $1 - x \geq 0$ to get

$$x E_1(1, Y^*) = x(1\ 0)AY^{*T} = x(a_{11}y^* + a_{12}(1 - y^*)) \leq x E_1(X^*, Y^*)$$

and

$$(1 - x)E_1(2, Y^*) = (1 - x)(0\ 1)AY^{*T}$$
$$= (1 - x)(a_{21}y^* + a_{22}(1 - y^*)) \leq (1 - x)E_1(X^*, Y^*).$$

Add these up to get

$$xE_{\mathrm{I}}(1, Y^*) + (1-x)E_{\mathrm{I}}(2, Y^*) = x(a_{11}y^* + a_{12}(1-y^*))$$
$$+ (1-x)(a_{21}y^* + a_{22}(1-y^*))$$
$$\leq xE_{\mathrm{I}}(X^*, Y^*) + (1-x)E_{\mathrm{I}}(X^*, Y^*)$$
$$= E_{\mathrm{I}}(X^*, Y^*)$$

But then, since

$$x(a_{11}y^* + a_{12}(1-y^*)) + (1-x)(a_{21}y^* + a_{22}(1-y^*)) = XAY^{*^T},$$

we see that

$$(x, (1-x))AY^{*^T} = XAY^{*^T} = E_{\mathrm{I}}(X, Y^*) \leq E_{\mathrm{I}}(X^*, Y^*).$$

Since $X \in S_2$ is any old mixed strategy for player I, this gives the first part of the definition that (X^*, Y^*) is a Nash equilibrium. □

Rational Reaction Set

- **Definition 3.2.2** *Let $X = (x, 1-x), Y = (y, 1-y)$ be strategies, and set $f(x, y) = E_I(X, Y)$, and $g(x, y) = E_{II}(X, Y)$. The* **rational reaction set** *for player I is the set of points*

$$R_I = \{(x, y) \mid 0 \leq x, y \leq 1, \max_{0 \leq z \leq 1} f(z, y) = f(x, y)\},$$

and the rational reaction set for player II is the set

$$R_{II} = \{(x, y) \mid 0 \leq x, y \leq 1, \max_{0 \leq w \leq 1} g(x, w) = g(x, y)\}.$$

  - (x^*, y^*) in both R_I and R_{II} says that $X^* = (x^*, 1 - x^*)$ and $Y^* = (y^*, 1 - y^*)$ is a Nash equilibrium.
  - Following to simplify notation, we drop the star on *X** and *Y** so that they will be simply $(x, 1-x), (y, 1-y)$, and assumed to be a Nash equilibrium.

Proposition 3.2.1 (1)-(2)

- For $E_{\mathrm{I}}(1, Y) \leq E_{\mathrm{I}}(X, Y)$ and $E_{\mathrm{I}}(2, Y) \leq E_{\mathrm{I}}(X, Y)$
  - We have the inequalities

  $$(a_{11} - a_{12} - a_{21} + a_{22})y + (a_{12} - a_{22}) + (a_{21} - a_{22})y + a_{22}$$
  $$\leq (a_{11} - a_{12} - a_{21} + a_{22})xy + (a_{12} - a_{22})x + (a_{21} - a_{22})y + a_{22}$$

  and

  $$(a_{11} - a_{12} - a_{21} + a_{22})xy + (a_{12} - a_{22})x + (a_{21} - a_{22})y + a_{22}$$
  $$\geq (a_{21} - a_{22})y + a_{22}.$$

  - Simplifying these two, we get

  $$M(1 - x)y - m(1 - x) \leq 0 \quad \text{and} \quad Mxy - mx \geq 0, \qquad (3.2.1)$$

  where $M = a_{11} - a_{12} - a_{21} + a_{22}$ and $m = a_{22} - a_{12}$.

Proposition 3.2.1 (1)-(2) (cont'd)

- Consider the following cases:

1. $M = m = 0$. In this case $M(1-x)y - m(1-x) = 0$, and $Mxy - mx = 0$, for any $x, y \in [0, 1]$. This is the trivial case because if $M = m = 0$, then $a_{22} = a_{12}$ and $a_{11} = a_{21}$. So it doesn't matter what player I does.

2. $M = 0, m > 0$. Then $-m(1-x) \le 0$ and $-mx \ge 0$, implying that $x = 0$ and y is anything in $[0, 1]$.

3. $M = 0, m < 0$. Then $(1-x) \le 0$, and $x \ge 0$. Solutions are $x = 1, 0 \le y \le 1$.

4. $M > 0$. Then $M(1-x)y - m(1-x) \leq 0$, and $Mxy - mx \geq 0$, and there are many solutions of these:

$$\text{if } x = 0 \implies 0 \leq y \leq \frac{m}{M},$$

$$\text{if } 0 < x < 1 \implies y = \frac{m}{M},$$

$$\text{if } x = 1 \implies 1 \geq y \geq \frac{m}{M}.$$

To see this, if $x = 1$, then $M(1-x)y - m(1-x) = 0 \leq 0$, and $My - m \geq 0$. So $y \geq m/M$. If $x = 0$, then $M(1-x)y - m(1-x) = My - m \leq 0$, and $Mxy - mx = 0 \geq 0$, so that $y \leq m/M$. If $0 < x < 1$, then $M(1-x)y - m(1-x) \leq 0 \implies My - m \leq 0$ and $Mxy - mx \geq 0 \implies My - m \geq 0$. Consequently $y = m/M$.

Proposition 3.2.1 (1)-(2) (cont'd)

5. $M < 0$. Then again $M(1-x)y - m(1-x) \leq 0$, and $Mxy - mx \geq 0$, and we have multiple solutions:
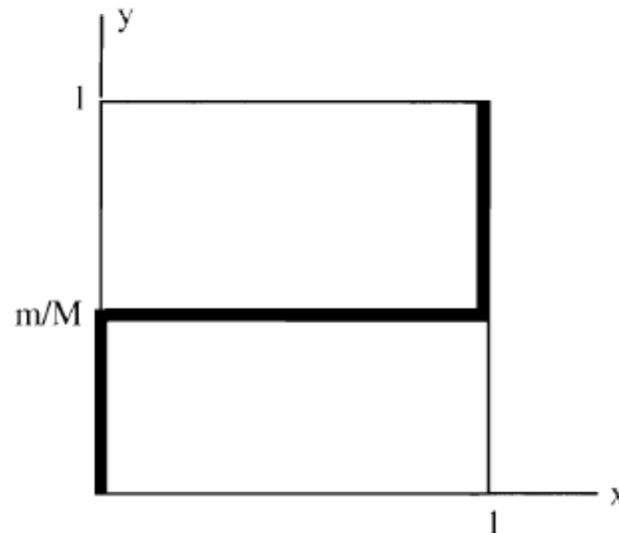
$$\text{if } x = 0 \implies 1 \geq y \geq \frac{m}{M},$$

$$\text{if } 0 < x < 1 \implies y = \frac{m}{M},$$

$$\text{if } x = 1 \implies 0 \leq y \leq \frac{m}{M}.$$

Proposition 3.2.1 (1)-(2) (cont'd)

- Figure 3.1 is a graph of the **set of the possible solutions**.
    - The bold zigzag line is the <u>rational reaction set for player I for a given *Y*</u>.
    - Expression of rational reaction set for player I in the case *M* > 0:

$$R_{\mathrm{I}} = \left\{(0, y) \mid 0 \le y \le \frac{m}{M}\right\} \cup \left\{\left(x, \frac{m}{M}\right) \mid 0 < x < 1\right\} \cup \left\{(1, y) \mid \frac{m}{M} \le y \le 1\right\}.$$

Looking for Nash: the case M>0

Figure 3.1   Rational reaction set for player I.

Proposition 3.2.1 (3)-(4)

- For $E_{II}(X^*, 1) \leq E_{II}(X^*, Y^*)$ and $E_{II}(X^*, 2) \leq E_{II}(X^*, Y^*)$
  - Similarly, we let
  $$R = b_{11} - b_{12} - b_{21} + b_{22}, \quad r = b_{22} - b_{21}.$$
  Then the inequalities we have to solve become
  $$Rx(1-y) - r(1-y) \leq 0, \quad Rxy - ry \geq 0.$$

- Consider the following cases:

1. $R = 0, r = 0$. Solutions are all $0 \leq x \leq 1, 0 \leq y \leq 1$.

2. $R = 0, r > 0$. Solutions are $0 \leq x \leq 1, y = 0$.

3. $R = 0, r < 0$. Solutions are $0 \leq x \leq 1, y = 1$.

Proposition 3.2.1 (3)-(4) (cont'd)

4. $R > 0$. Solutions are

$$\text{if } y = 0 \implies 0 \le x \le \frac{r}{R},$$

$$\text{if } 0 < y < 1 \implies x = \frac{r}{R},$$

$$\text{if } y = 1 \implies 1 \ge x \ge \frac{r}{R}.$$

5. $R < 0$. In this final case the set of all possible solutions are

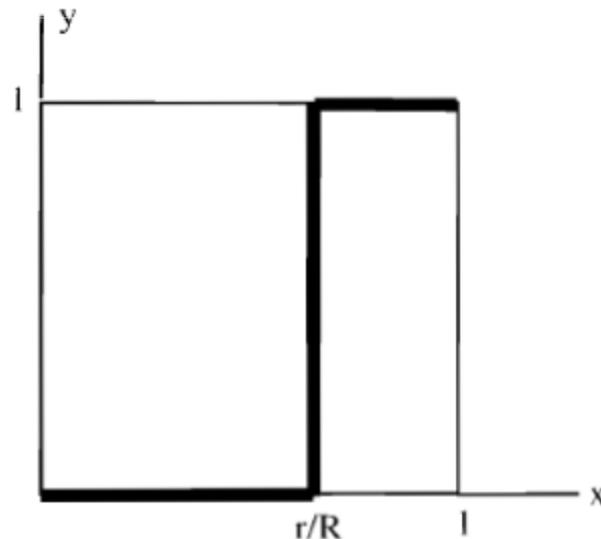$$\text{if } y = 0 \implies 0 \le x \le \frac{r}{R},$$

$$\text{if } 0 < y < 1 \implies x = \frac{r}{R},$$

$$\text{if } y = 1 \implies 1 \ge x \ge \frac{r}{R}.$$

Proposition 3.2.1 (3)-(4) (cont'd)

- Figure 3.2 is a graph of the set of the possible solutions.
  - Bold zigzag line is the <u>rational reaction set for player II for a given *X*.</u>
  - Expression of rational reaction set for player II in the case R < 0:

$$R_{\text{II}} = \left\{ (x,0) \mid 0 \le x \le \frac{r}{R} \right\} \cup \left\{ \left(\frac{r}{R}, y \right) \mid 0 < y < 1 \right\} \cup \left\{ (x,1) \mid \frac{r}{R} \le x \le 1 \right\}.$$


Looking for Nash: the case R<0

Figure 3.2 Rational reaction set for player II.

Rational Reaction Set (cont'd)

- Rational reaction set for both players
  - If we lay the graph in Figure 3.2 of R_{II} on top of that in Figure 3.1 of the set R_I, the point of intersection in $R_I \cap R_{II}$ is the Nash equilibrium.
  - The mixed Nash equilibrium is at the point which is in both rational reaction sets for each player.
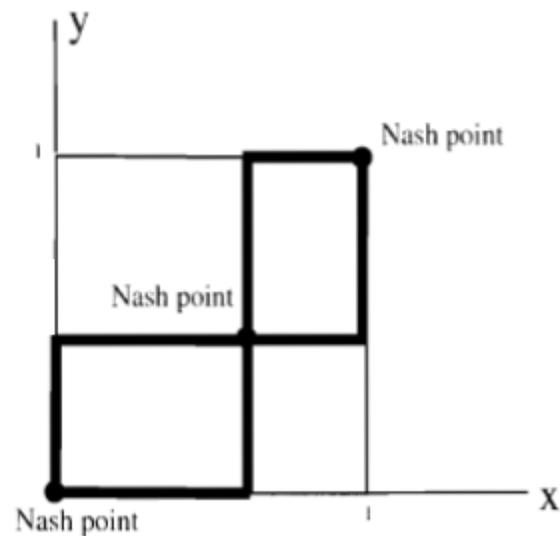
The Mixed Nash Equilibrium

- In the case $M \neq 0, R \neq 0,$ we have a mixed Nash equilibrium $X^* = (x, 1 - x), Y^* = (y, 1 - y)$ at

$$x = \frac{r}{R} = \frac{a_{22} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}} \quad \text{and} \quad y = \frac{m}{M} = \frac{b_{22} - b_{21}}{b_{11} - b_{12} - b_{21} + b_{22}}.$$

  - The pure Nash equilibria will be the intersection points of the rational reaction sets at the corners.
  - The expected payoffs to each player are calculated after determination of the Nash equilibria by calculating $X^* A Y^{*T}$ and $X^* B Y^{*T}$.

Example 3.5

- The bimatrix game with the two matrices

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

- The pair (2,1) and (1,2),which the first number is the largest in the first column and the second number is the largest in the first row in (A,B).
- $X^* = (1,0), Y^* = (1,0)$ is a Nash point as is $X^* = (0,1), Y^* = (0,1)$.
- Apply the solution results obtained in the theorem, then

$$M = 2 - (-1) - (-1) + 1 = 5 > 0, m = 1 - (-1) = 2, \frac{m}{M} = \frac{2}{5},$$

$$R = 5, r = 3, \frac{r}{R} = \frac{3}{5},$$

and we have three equilibria at points $(x, y) = (0,0), (x, y) = (\frac{3}{5}, \frac{2}{5})$, and $(x, y) = (1, 1)$.

Example 3.5 (cont'd)

- In Figure 3.3, the three equilibria are where the two zigzag lines cross.
- Pure Nash equilibria: $(x, y) = (0, 0),$ and $(1, 1)$ (on the boundary).
- Mixed Nash equilibrium: $(x, y) = (\frac{3}{5}, \frac{2}{5})$ (in the interior).
- The expected payoffs are $E_{\mathrm{I}}(X^*, Y^*) = \dfrac{1}{5}$ and $E_{\mathrm{II}}(X^*, Y^*) = \dfrac{1}{5}.$
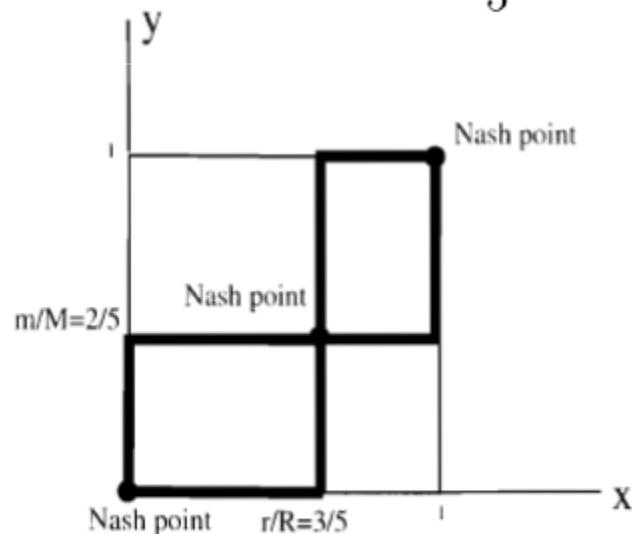
Nash Equilibria

Figure 3.3 Rational reaction sets for both players

Rational Reaction Sets Calculation

- **Remark: A direct way to calculate the rational reaction sets for 2 x 2 games.**

  - Let $X = (x, 1-x)$, $Y = (y, 1-y)$ be any strategies and define

  $$f(x,y) = E_{\text{I}}(X,Y) \text{ and } g(x,y) = E_{\text{II}}(X,Y).$$

  - The idea is to find for a fixed $0 \le y \le 1$, the best response to y. Accordingly,

  $$\max_{0 \le x \le 1} f(x,y) = \max_{0 \le x \le 1} xE(1,Y) + (1-x)E_{\text{I}}(2,Y)$$

  $$= x[E_{\text{I}}(1,Y) - E_{\text{I}}(2,Y)] + E_{\text{I}}(2,Y)$$

  $$= \begin{cases} E_{\text{I}}(2,Y) & \text{at } x = 0 \text{ if } E_{\text{I}}(1,Y) < E_{\text{I}}(2,Y); \\ E_{\text{I}}(1,Y) & \text{at } x = 1 \text{ if } E_{\text{I}}(1,Y) > E_{\text{I}}(2,Y); \\ E_{\text{I}}(2,Y) & \text{at any } 0 < x < 1 \text{ if } E_{\text{I}}(1,Y) = E_{\text{I}}(2,Y). \end{cases}$$

Rational Reaction Sets Calculation (cont'd)

– For example,

$$E_I(1, Y) < E_I(2, Y) \Leftrightarrow My < m,$$

$$M = a_{11} - a_{12} - a_{21} + a_{22}, \ m = a_{22} - a_{12}.$$

If $M > 0$ this is equivalent to the condition $0 \leq y < m/M$. Consequently, in the case $M > 0$, the best response to any $0 \leq y < M/m$ is $x = 0$.

Requirement of the Inequalities of a
Nash Equilibrium

- **Proposition 3.2.3** (X^*, Y^*) *is a Nash equilibrium if and only if*

$$E_I(i, Y^*) = {}_iAY^{*T} \leq X^*AY^{*T} = E_I(X^*, Y^*), \quad i = 1, \ldots, n,$$

$$E_{II}(X^*, j) = X^*B_j \leq X^*BY^{*T} = E_{II}(X^*, Y^*), \quad j = 1, \ldots, m.$$

Proof. If $E_I(i, Y^*) = {}_iAY^{*T} \leq X^*AY^{*T} \equiv v_A$, for all rows, then we take any $X = (x_1, \ldots, x_n) \in S_n$. By multiplying and adding, we obtain

$$\sum_{i=1}^{n} x_i E_I(i, Y^*) \leq \sum_{i=1}^{n} x_i v_A = v_A.$$

But the left side of this inequality is $E_I(X, Y^*) = XAY^{*T}$ and so $E_I(X, Y^*) \leq E_I(X^*, Y^*)$, for any $X \in S_n$. The remaining parts of this claim follow in the same way. $\square$

Example 3.6

- Someone says that the bimatrix game

$$\begin{bmatrix} (2,1) & (-1,-1) \\ (-1,-1) & (1,2) \end{bmatrix}$$

has a Nash equilibrium at $X^* = (\frac{3}{5}, \frac{2}{5}), Y^* = (\frac{2}{5}, \frac{3}{5})$.

- To check that, first compute

$$E_{\mathrm{I}}(X^*, Y^*) = E_{\mathrm{II}}(X^*, Y^*) = \tfrac{1}{5}.$$

- Next check that this number is at least as good as what could be gained if the other player plays a pure strategy.
- In fact, $E_{\mathrm{I}}(1, Y^*) = \frac{1}{5} = E_{\mathrm{I}}(2, Y^*)$ and also $E_{\mathrm{II}}(X^*, 1) = E_{\mathrm{II}}(X^*, 2) = \frac{1}{5}$, so we do indeed have a Nash point.

Equality of Payoffs Theorem

- **Theorem 3.2.4 (*Equality of Payoffs Theorem*)** *Suppose that*

$$X^* = (x_1, x_2, \ldots, x_n), \quad Y^* = (y_1, y_2, \ldots, y_m)$$

is a Nash equilibrium for the bimatrix game (A, B).

For any row k *that has a positive probability of being used,* $x_k > 0$, *we have* $E_I(k, Y^*) = E_I(X^*, Y^*) \equiv v_I$.

For any column j *that has a positive probability of being used,* $y_j > 0$, *we have* $E_{II}(X^*, j) = E_{II}(X^*, Y^*) \equiv v_{II}$. *That is,*

$$x_k > 0 \implies E_I(k, Y^*) = v_I$$
$$y_j > 0 \implies E_{II}(X^*, j) = v_{II}.$$

Equality of Payoffs Theorem (cont'd)

Proof. We know that since we have a Nash point, $E_I(X^*, Y^*) = v_I \geq E_I(i, Y^*)$ for any row i. Now, suppose that row k has positive probability of being played against Y^* and that it gives player I a strictly smaller expected payoff $v_I > E_I(k, Y^*)$. Then $v_I \geq E_I(i, Y^*)$ for all the rows $i = 1, 2, \ldots, n, i \neq k$, and $v_I > E_I(k, Y^*)$ together imply that

$$x_i v_I \geq x_i E_I(i, Y^*), i \neq k, \text{ and } x_k v_I > x_k E_I(k, Y^*).$$

Adding up all these inequalities, we get

$$\sum_{i=1}^{n} x_i v_I = v_I > \sum_{i=1}^{n} x_i E_I(i, Y^*) = E_I(X^*, Y^*) = v_I.$$

This contradiction says it must be true that $v_I = E_I(k, Y^*)$. The only thing that could have gone wrong with this argument is $x_k = 0$.

Equality of Payoffs Theorem (cont'd)

- We can find the (completely) mixed Nash equilibria by solving a system of equations rather than inequalities for player II:

$$_k AY^{*T} = E_I(k, Y^*) = E_I(s, Y^*) = {}_s AY^{*T}, \text{ assuming that } x_k > 0, x_s > 0,$$

and

$$X^* B_j = E_{II}(X^*, j) = E_{II}(X^*, r) = X^* B_r, \text{ assuming that } y_j > 0, y_r > 0.$$

Also with the additional condition

$$x_1 + x_2 + \cdots + x_n = 1, \quad y_1 + y_2 + \cdots + y_m = 1.$$

Example 3.7

- Consider the matrices with $0 < x_1 < 1$,

$$A = \begin{bmatrix} -4 & 2 \\ 2 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}.$$

  - By the equality of payoffs Theorem 3.2.4, $v_{\mathrm{I}} = E_{\mathrm{I}}(1, Y) = E_{\mathrm{I}}(2, Y)$,
    we have $\quad 2y_1 + y_2 = -4y_1 + 2y_2, \quad$ and $\; y_1 + y_2 = 1$
    and get $\quad y_1 = 0.143, \; y_2 = 0.857,$ and $v_{\mathrm{I}} = 1.143.$
  - Similarly, $E_{\mathrm{II}}(X, 1) = E_{\mathrm{II}}(X, 2)$ gives
    $x_1 + 2x_2 = 3x_2 \quad$ and $\; x_1 + x_2 = 1 \implies x_1 = x_2 = 0.5, v_{\mathrm{II}} = 1.5.$
  - Notice that we can find the Nash point without actually knowing v_{I} or v_{II}.
    - The Nash point for II is found from the payoff function for player I and vice versa.

Interior Mixed Nash Points by Calculus

Calculus Method for Interior Nash (3.3.1)

1. The payoff matrices are $A_{n \times m}$ for player I and $B_{n \times m}$ for player II. The expected payoff to I is $E_I(X, Y) = X A Y^T$, and the expected payoff to II is $E_{II}(X, Y) = X B Y^T$.

2. Let $x_n = 1 - (x_1 + \cdots + x_{n-1}) = x_n - \sum_{i=1}^n x_i$, $y_m = 1 - \sum_{j=1}^{m-1} y_j$ so each expected payoff is a function only of $x_1, \ldots, x_{n-1}, y_1, \ldots, y_{m-1}$. We can write

$$E_1(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{m-1}) = E_I(X, Y),$$
$$E_2(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{m-1}) = E_{II}(X, Y).$$

3. Take the partial derivatives and solve the system of equations $\partial E_1 / \partial x_i = 0$, $\partial E_2 / \partial y_j = 0$, $i = 1, \ldots, n-1$, $j = 1, \ldots, m-1$.

4. If there is a solution of this system of equations which satisfies the constraints $x_i \geq 0, y_j \geq 0$ and $\sum_{i=1}^{n-1} x_i \leq 1, \sum_{j=1}^{m-1} y_j \leq 1$, then this is the mixed strategy Nash equilibrium.

Calculus Method for Interior Nash (3.3.1) (cont'd)

- Remark
  - We do **not** maximize $E_1(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{m-1})$ over **all** variables x and y, but only over the x variables. Similarly, we do **not** maximize $E_2(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{m-1})$ over **all** variables x and y, but only over the y variables.
  - Apply calculus will give us all the interior, that is, completely mixed Nash points.
  - Calculus cannot give us the pure Nash equilibria because those are achieved on the boundary of the strategy region.

Example 3.8

- Use the calculus method to solve the game in the preceding section with matrices

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

- First, set up the functions (using $X = (x, 1 - x)$, $Y = (y, 1 - y)$)

$$E_1(x, y) = [2x - (1 - x)]y + [-x + (1 - x)](1 - y),$$
$$E_2(x, y) = [x - (1 - x)]y + [-x + 2(1 - x)](1 - y).$$

- Player I wants to maximize E_1 for each fixed y, so we take

$$\frac{\partial E_1(x, y)}{\partial x} = 3y + 2y - 2 = 5y - 2 = 0 \implies y = \frac{2}{5}.$$

Example 3.8 (cont'd)

– Similarly, player II wants to maximize $E_2(x, y)$ for each fixed x, so

$$\frac{\partial E_2(x, y)}{\partial y} = 5x - 3 = 0 \implies x = \frac{3}{5}.$$

– Everything works to give us $X^* = (\frac{3}{5}, \frac{2}{5})$ and $Y^* = (\frac{2}{5}, \frac{3}{5})$ is a Nash equilibrium for the game, just as we had before.

– Notice that we do not get the pure Nash points for this problem.

Example 3.9

- Two partners have two choices for where to invest their money, say, O_1, O_2 where the letter O stands for opportunity, but they have to come to an agreement. We model this using the bimatrix

$$\begin{array}{c|cc} & O_1 & O_2 \\ \hline O_1 & (1,2) & (0,0) \\ O_2 & (0,0) & (2,1) \end{array}$$
.

  - There are two pure Nash points at (O_1, O_1) and (O_2, O_2).
  - We will start the analysis from the beginning rather than using the formulas from section 3.2.
  - We will derive the rational reaction sets for each player directly. Set

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Example 3.9 (cont'd)

— For player I

$$E_1(x, y) = (x \ 2(1-x)) \cdot (y, 1-y)^T$$
$$= xy + 2(1-x)(1-y) = 3xy - 2x - 2y + 2$$
$$= x(3y - 2) - 2y + 2.$$

$$\max_{0 \le x \le 1} E_1(x, y) = \begin{cases} y & \text{if } 3y - 2 > 0 \implies y > \frac{2}{3}, \text{ achieved at } x = 1; \\ \frac{2}{3} & \text{if } y = \frac{2}{3}, \text{ achieved at any } x \in [0, 1]; \\ -2y + 2 & \text{if } 3y - 2 < 0 \implies y < \frac{2}{3}, \text{ achieved at } x = 0. \end{cases}$$

So the rational reaction set for player I:

$$R_I = \{(x^*, y) \in [0, 1] \times [0, 1] \mid \max_{0 \le x \le 1} E_1(x, y) = E_1(x^*, y)\}$$
$$= \left\{ (1, y), \frac{2}{3} < y \le 1 \right\} \cup \left\{ \left(x, \frac{2}{3} \right), 0 \le x \le 1 \right\} \cup \left\{ (0, y), 0 \le y < \frac{2}{3} \right\}.$$

Example 3.9 (cont'd)

- For player II

$$E_2(x, y) = 2xy + (1 - x)(1 - y) = 3xy - x - y + 1 = y(3x - 1) - x + 1.$$

$$\max_{y \in [0,1]} E_2(x, y) = \begin{cases} -x + 1 & \text{if } 0 \leq x < \frac{1}{3} \text{ achieved at } y = 0; \\ \frac{2}{3} & \text{if } x = \frac{1}{3} \text{ achieved at any } y \in [0, 1] \\ 2x & \text{if } \frac{1}{3} < x \leq 1 \text{ achieved at } y = 1. \end{cases}$$

The rational reaction set for player II:

$$R_{\text{II}} = \{(x, y^*) \in [0, 1] \times [0, 1] \mid \max_{0 \leq y \leq 1} E_2(x, y) = E_2(x, y^*)\}$$

$$= \left\{(x, 0), 0 \leq x < \frac{1}{3}\right\} \cup \left\{\left(\frac{1}{3}, y\right), 0 \leq y \leq 1\right\} \cup \left\{(x, 1), \frac{1}{3} < x \leq 1\right\}.$$

Example 3.9 (cont'd)

- Rational reaction sets for both players (figure below)
  - Notice that the rational reaction sets and graphs do **not** indicate what the **payoffs** are to the individual players, but only their **strategies**.
  - The zigzag lines cross (which is the set of points $R_I \cap R_{II}$) are all the Nash points: $(x, y) = (0, 0), (1, 1)$ and $\left(\frac{1}{3}, \frac{2}{3}\right).$
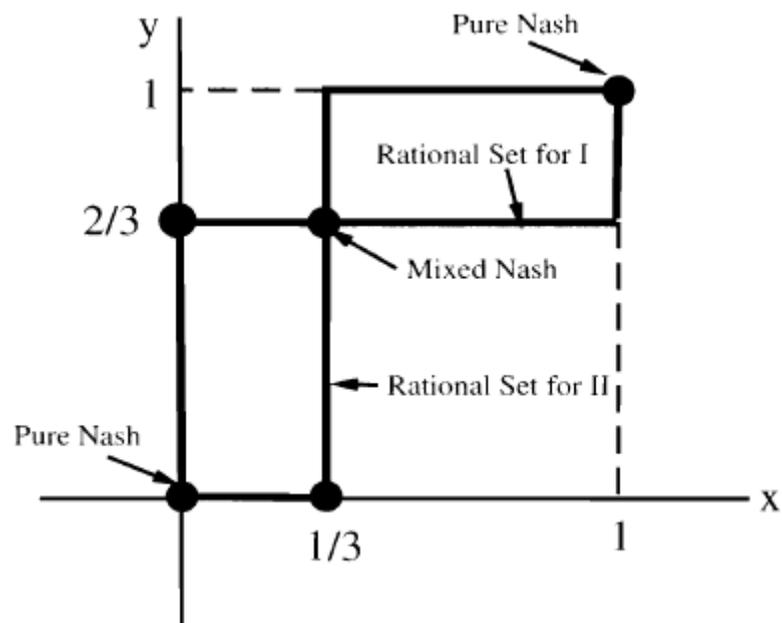
Example 3.9 (cont'd)

– The associated expected payoffs are

$$E_1(0,0) = 2, \quad E_2(0,0) = 1,$$

$$E_1(1,1) = 1, \quad E_2(1,1) = 2,$$

and

$$E_1\left(\frac{1}{3}, \frac{2}{3}\right) = \frac{2}{3} = E_2\left(\frac{1}{3}, \frac{2}{3}\right).$$

– Only the mixed strategy Nash point $(X^*, Y^*) = ((\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, \frac{1}{3}))$ gives the same expected payoffs to the two players—**fair** but **less**.

– Calculus will give us the interior mixed Nash very easily:

$$\frac{\partial E_1(x,y)}{\partial x} = 3y - 2 = 0 \implies y = \frac{2}{3}, \quad \text{and}$$

$$\frac{\partial E_2(x,y)}{\partial y} = 3x - 1 = 0 \implies x = \frac{1}{3}.$$

Definition of Rational Reaction Sets

- **Definition 3.3.1** *The rational reaction sets for each player are defined as follows:*

$$R_I = \{(X,Y) \in S_n \times S_m \mid E_I(X,Y) = \max_{p \in S_n} E_I(p,Y)\},$$

$$R_{II} = \{(X,Y) \in S_n \times S_m \mid E_{II}(X,Y) = \max_{t \in S_m} E_{II}(X,t)\}.$$

The set of all Nash equilibria is then the set of all common points $R_I \cap R_{II}$.

  - The definition above is the rational reaction sets in the general case with arbitrary size matrices.

Equations for an Interior Nash Equilibrium

- We can write down the system of equations that we get using calculus in the general case. The process is as follows:

  - Start with

$$E_1(X,Y) = XAY^T = \sum_{j=1}^{m}\sum_{i=1}^{n} x_i a_{ij} y_j.$$

  - Following the calculus method (3.3.1) with $x_n = 1 - \sum_{k=1}^{n-1} x_k$:

$$
\begin{aligned}
XAY^T &= \sum_{j=1}^{m}\sum_{i=1}^{n} x_i a_{ij} y_j = \sum_{j=1}^{m}\left(\sum_{i=1}^{n-1} x_i a_{ij} y_j + \left(1 - \sum_{k=1}^{n-1} x_k\right) a_{nj} y_j\right) \\
&= \sum_{j=1}^{m}\left(a_{nj} y_j + \sum_{i=1}^{n-1} x_i a_{ij} y_j - \sum_{k=1}^{n-1} x_k(a_{nj} y_j)\right) \\
&= \sum_{j=1}^{m}\left(a_{nj} y_j + \sum_{i=1}^{n-1} x_i[a_{ij} - a_{nj}] y_j\right) = E_1(x_1,\ldots,x_{n-1};y_1,\ldots,y_m).
\end{aligned}
$$

Equations for an Interior Nash Equilibrium (cont'd)

- But then, for each $k = 1, 2, \ldots, n-1$, we obtain

$$\frac{\partial E_1(x_1, \ldots, x_{n-1}, y_1, \ldots, y_m)}{\partial x_k} = \sum_{j=1}^{m} y_j [a_{kj} - a_{nj}].$$

- Similarly, for each $s = 1, 2, \ldots, m-1$, we get the partials

$$\frac{\partial E_2(x_1, \ldots, x_n, y_1, \ldots, y_{m-1})}{\partial y_s} = \sum_{i=1}^{n} x_i [b_{is} - b_{im}].$$

Equations for an Interior Nash Equilibrium (cont'd)

- So, the system of equations we need to solve to get an interior Nash equilibrium is

$$\left.\begin{array}{ll} \sum_{j=1}^{m} y_j[a_{kj} - a_{nj}] = 0, & k = 1, 2, \ldots, n-1, \\[2em] \sum_{i=1}^{n} x_i[b_{is} - b_{im}] = 0, & s = 1, 2, \ldots, m-1, \\[2em] x_n = 1 - \sum_{i=1}^{n-1} x_i, \quad y_m = 1 - \sum_{j=1}^{m-1} y_j. \end{array}\right\} \qquad (3.3.2)$$

- Once these are solved, we check that $x_i \geq 0, y_j \geq 0$ and if so we get the Nash equilibrium $X^* = (x_1, \ldots, x_n)$ and $Y^* = (y_1, \ldots, y_m)$.

Equations for an Interior Nash Equilibrium (cont'd)

- Notice that the equations are really two separate systems of linear equations and can be solved separately.
  - The variables x_i and y_j appear only in their own system.
- Also notice that these equations are really nothing more than the equality of payoffs Theorem 3.2.4. For example,

$$\sum_{j=1}^{m} y_j[a_{kj} - a_{nj}] = 0 \implies \sum_{j=1}^{m} y_j a_{kj} = \sum_{j=1}^{m} y_j a_{nj},$$

which is the same as saying that for $k = 1, 2, \ldots, n-1$, we have

$$E_{\mathrm{I}}(k, Y^*) = \sum_{j=1}^{m} y_j a_{kj} = \sum_{j=1}^{m} y_j a_{nj} = E_{\mathrm{I}}(n, Y^*).$$

- These equations won't necessarily work for the pure Nash or the ones with zero components.

Example 3.10

- Use the equations (3.3.2) to find interior Nash points for the following bimatrix game:

$$A = \begin{bmatrix} -2 & 5 & 1 \\ -3 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} -4 & -2 & 4 \\ -3 & 1 & 4 \\ 3 & 1 & -1 \end{bmatrix}.$$

  – By the system of equations (3.3.2), we have

$$\begin{cases} -2y_1 + 6y_2 - 2 = 0, \quad -5y_1 + y_2 = 0 & \Longrightarrow \quad y_1 = \frac{1}{14}, y_2 = \frac{5}{14} \\ -12x_1 - 11x_2 + 4 = 0, \quad -8x_1 - 5x_2 + 2 = 0 & \Longrightarrow \quad x_1 = \frac{1}{14}, x_2 = \frac{4}{14} \end{cases}$$

Besides, $\sum_j y_j = 1$ and $\sum_i x_i = 1$ $\qquad \Longrightarrow \quad y_3 = \frac{8}{14}, x_3 = \frac{9}{14}$

Example 3.10 (cont'd)

– So the interior Nash point is

$$X^* = \left(\frac{1}{14}, \frac{4}{14}, \frac{9}{14} \right) \text{ and } Y^* = \left(\frac{1}{14}, \frac{5}{14}, \frac{8}{14} \right).$$

The expected payoffs to each player are

$$E_{\mathrm{I}}(X^*, Y^*) = X^* A Y^{*T} = \frac{31}{14}$$

$$E_{\mathrm{II}}(X^*, Y^*) = X^* B Y^{*T} = \frac{11}{14}.$$

– There are also two pure Nash points:

$$X^* = (0, 0, 1), Y^* = (1, 0, 0) \text{ with payoffs } (2, 3)$$

$$X^* = (0, 1, 0), Y^* = (0, 0, 1) \text{ with payoffs } (3, 4).$$

Example 3.10 (cont'd)

- Maple can be used to solve the system of equations giving an interior Nash point.

```
> restart:with(LinearAlgebra):
> A:=Matrix([[-2,5,1],[-3,2,3],[2,1,3]]);
> B:=Matrix([[-4,-2,4],[-3,1,4],[3,1,-1]]);
> Y:=Vector(3,symbol=y):
> X:=Vector(3,symbol=x):
> yeq:=seq(add(y[j]*(A[i,j]-A[3,j]),j=1..3),i=1..2);
> xeq:=seq(add(x[i]*(B[i,s]-B[i,3]),i=1..3),s=1..2);
> xsols:=solve({xeq[1]=0,xeq[2]=0,add(x[i],i=1..3)=1},
                        [x[1],x[2],x[3]]);
> assign(xsols);
> ysols:=solve({yeq[1]=0,yeq[2]=0,add(y[j],j=1..3)=1},
                        [y[1],y[2],y[3]]);
> assign(ysols);
> Transpose(X).A.Y;  Transpose(X).B.Y;
```

Example 3.10 (cont'd)

- Remark of the Maple command
  - In Maple, vectors are defined as **column** matrices, so a correct multiplication is as shown in the Maple commands in the last line, even though in the book we use $X A Y^T$.
  - If you change some of the numbers in the matrices A, B and rerun the Maple code, you will see that frequently the solutions will have negative components or the components will be greater than one.
    - Especially if there is more than one interior Nash equilibrium (which could occur for matrices that have more than two pure strategies).

Example 3.11

- Consider a example in which the <u>equations do not work</u> (see problem 3.11) because it turns out that one of the columns should never be played by player II.

  - The <u>mixed Nash is not in the interior</u>, but on the boundary of $S_n \times S_m$.

  - Let's consider the game with payoff matrices

  $$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 4 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}.$$

  - After calculus, we have

  $value(A) = 2,$ with pure saddle $X_A = (1, 0), Y_A = (1, 0, 0),$
  $value(B^T) = 1,$ with saddle $X_B = (0, 0, 1), Y_B = (\frac{1}{2}, \frac{1}{2}).$

  Now, let $X = (x, 1-x), Y = (y_1, y_2, 1 - y_1 - y_2)$ be a Nash point .

Example 3.11 (cont'd)

– For player I

$$E_1(x, y_1, y_2) = XAY^T = x[y_1 - 2y_2 + 1] + y_2 - 3y_1 + 3.$$

$$\max_{0 \leq x \leq 1} x[y_1 - 2y_2 + 1] \quad + y_2 - 3y_1 + 3$$

$$= \begin{cases} -2y_1 - y_2 + 4 & \text{if } y_1 > 2y_2 - 1; \\ y_2 - 3y_1 + 3 & \text{if } y_1 = 2y_2 - 1; \\ y_2 - 3y_1 + 3 & \text{if } y_1 < 2y_2 - 1. \end{cases} = \begin{cases} E_1(1, y_1, y_2) & \text{if } y_1 > 2y_2 - 1; \\ E_1(x, 2y_2 - 1, y_2) & \text{if } y_1 = 2y_2 - 1; \\ E_1(0, y_1, y_2) & \text{if } y_1 < 2y_2 - 1. \end{cases}$$

Along any point of the straight line $y_1 = 2y_2 - 1$ the maximum of $E_1(x, y_1, y_2)$ is achieved at any point $0 \leq x \leq 1$.

Finally, the rational reaction set for player I is

$$R_{\text{I}} = \{(x, y_1, y_2) \mid [(1, y_1, y_2), y_1 > 2y_2 - 1], \text{ or}$$

$$[(x, 2y_2 - 1, y_2), 0 \leq x \leq 1], \text{ or } [(0, y_1, y_2), y_1 < 2y_2 - 1]\}.$$

Example 3.11 (cont'd)

- For player II

$$E_2(x, y_1, y_2) = XBY^T = y_1 + y_2(3x - 1) + (-2x + 1).$$

$$\max_{\substack{y_1+y_2 \leq 1. \\ y_1, y_2 \geq 0}} y_1 + y_2(3x - 1) + (-2x + 1)$$

$$= \begin{cases} -2x + 2 & \text{if } 3x - 1 < 1; \\ \frac{2}{3} & \text{if } 3x - 1 = 1; \\ x & \text{if } 3x - 1 > 1. \end{cases} = \begin{cases} E_2(x, 1, 0) & \text{if } 0 \leq x < \frac{2}{3}; \\ E_2\left(\frac{2}{3}, y_1, y_2\right) = \frac{2}{3} & \text{if } x = \frac{2}{3}, \ y_1 + y_2 = 1; \\ E_2(x, 0, 1) & \text{if } \frac{2}{3} < x \leq 1. \end{cases}$$

The rational reaction set for player II is

$$R_{\text{II}} = \left\{ (x, y_1, y_2) \mid \left[(x, 1, 0), 0 \leq x < \frac{2}{3} \right], \text{ or } \right.$$

$$\left. \left[\left(\frac{2}{3}, y_1, y_2\right), y_1 + y_2 = 1 \right], \text{ or } \left[(x, 0, 1), \frac{2}{3} < x \leq 1 \right] \right\}.$$

Example 3.11 (cont'd)

- The graph of R_I and R_{II} on the same graph (in three dimensions) will intersect at the mixed Nash equilibrium points.

- Nash equilibrium

$$X^* = \left(\frac{2}{3}, \frac{1}{3}\right), \quad Y^* = \left(\frac{1}{3}, \frac{2}{3}, 0\right).$$

- Expected payoff

$$E_1\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right) = \frac{8}{3} \text{ and } E_2\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right) = \frac{2}{3}.$$

- We could have simplified the calculations by the fact that column 3 for player II is dominated by column 1.

Nash's Theorem

- We will show that the set of all Nash equilibria is then the set of all common points $R_{\mathrm{I}} \cap R_{\mathrm{II}}$.

    - **Theorem 3.3.2** *There exists $X^* \in S_n$ and $Y^* \in S_m$ so that*

    $$E_I(X^*, Y^*) = X^* A Y^{*T} \geq E_I(X, Y^*),$$
    $$E_{II}(X^*, Y^*) = X^* B Y^{*T} \geq E_{II}(X^*, Y),$$

    for any other mixed strategies $X \in S_n, Y \in S_m$.

    - The theorem guarantees at least one Nash equilibrium if we are willing to use mixed strategies.
    - We will give a proof that is very similar to that of von Neumann's theorem using the Kakutani fixed-point theorem for point to set maps.

Proof that There Is a Nash Equilibrium for Bimatrix Games

- **Proof.**

  First $S_n \times S_m$ is a closed, bounded and convex set. Now for each given pair of strategies (X, Y), we could consider the best response of player II to *X* and the best response of player I to *Y*.

  - **Definition 3.3.3** *The* **best response sets** *for each player are defined as*

  $$BR_I(Y) = \{X \in S_n \mid E_I(X, Y) = \max_{p \in S_n} E_I(p, Y)\},$$

  $$BR_{II}(X) = \{Y \in S_m \mid E_{II}(X, Y) = \max_{t \in S_m} E_I(X, t)\}.$$

  - The difference between the best response set and the rational reaction set is that the rational reaction set R_1 consists of the **pairs of strategies** (X, Y) for which $E_1(X, Y) = \max_p E_1(p, Y)$, whereas $BR_1(Y)$ consists of the strategy X for player I that is the best response to a **fixed** Y.

Proof that There Is a Nash Equilibrium for Bimatrix Games (cont'd)

- Whenever you are maximizing a continuous function, which is true of $E_{\mathrm{I}}(X, Y)$, over a closed and bounded set (which is true of S_n), you always have a point at which the maximum is achieved. So we know that $BR_{\mathrm{I}}(Y) \neq \emptyset$. Similarly, the same is true of $BR_{\mathrm{II}}(X) \neq \emptyset$.

  We define the point to set mapping

  $$\varphi : (X, Y) \in S_n \times S_m \rightarrow BR_{\mathrm{I}}(Y) \times BR_{\mathrm{II}}(X) \subset S_n \times S_m,$$

  which gives, for each pair (X, Y) of mixed strategies, the best response strategies $(X', Y') \in \varphi(X, Y)$ with $X' \in BR_{\mathrm{I}}(Y)$ and $Y' \in BR_{\mathrm{II}}(X)$.

Proof that There Is a Nash Equilibrium for Bimatrix Games (cont'd)

It seems natural that our Nash equilibrium should be among the best response strategies to the opponent. Translated, this means that a Nash equilibrium (X^*, Y^*) should satisfy $(X^*, Y^*) \in \varphi(X^*, Y^*)$. But that is exactly what it means to be a fixed point of φ. If φ satisfies the required properties to apply Kakutani's fixed-point theorem, we have the existence of a Nash equilibrium. This is relatively easy to check because $X \mapsto E_I(X, Y)$ and $X \mapsto E_{II}(X, Y)$ are linear maps, as are $Y \mapsto E_I(X, Y)$ and $Y \mapsto E_{II}(X, Y)$. Hence it is easy to show that $\varphi(X, Y)$ is a convex, closed, and bounded subset of $S_n \times S_m$. It is also not hard to show that φ will be an (upper) semicontinuous map, and so Kakutani's theorem applies.

This gives us a pair $(X^*, Y^*) \in \varphi(X^*, Y^*)$. Written out, this means $X^* \in BR_I(Y^*)$ so that

$$E_I(X^*, Y^*) = \max_{p \in S_n} E_I(p, Y^*) \geq E_I(X, Y^*) \text{ for all } X \in S_n$$

and $Y^* \in BR_{II}(X^*)$ so that

$$E_{II}(X^*, Y^*) = \max_{t \in S_m} E_{II}(X^*, t) \geq E_{II}(X^*, Y) \text{ for all } Y \in S_m.$$

That's it. (X^*, Y^*) is a Nash equilibrium. $\quad\square$

Remark

- It is important to understand the difficulty in obtaining the existence of a Nash equilibrium.
    - If our problem was

$$\max_{X \in S_n, Y \in S_m} E_{\mathrm{I}}(X,Y) \text{ and } \max_{X \in S_n, Y \in S_m} E_{\mathrm{II}}(X,Y),$$

      then the existence of an $(X_{\mathrm{I}}, Y_{\mathrm{I}})$ providing the maximum of E_{I} is immediate from the fact that $E_{\mathrm{I}}(X,Y)$ is a continuous function over a closed and bounded set. The same is true for the existence of an $(X_{\mathrm{II}}, Y_{\mathrm{II}})$ providing the maximum of $E_{\mathrm{II}}(X,Y)$.

Nonlinear Programming Method for Nonzero Sum Two-Person Games

Nonlinear Program

- A **nonlinear program** is a method of finding all Nash equilibria for arbitrary two-person nonzero sum games with any number of strategies.

    – For example, if we have an objective function f and constraint functions $h_1, \ldots, h_k$, the problem

$$\text{Minimize } f(x_1, \ldots, x_n) \quad \text{subject to } h_j(x_1, \ldots, x_n) \leq 0, j = 1, \ldots, k,$$

    is the general formulation of a nonlinear programming problem.

    – If the function f is quadratic and the constraint functions are linear, then this is called a **quadratic programming problem.**

    – Once we formulate the game as a nonlinear program, we will use the packages developed in Maple to solve them numerically.

Nonlinear Program (cont'd)

- **Theorem 3.4.1** *Consider the two-person game with matrices (A, B) for players I and II. Then, $(X^* \in S_n, Y^* \in S_m)$ is a Nash equilibrium if and only if they satisfy, along with scalars p^*, q^* the nonlinear program*

$$\max_{X,Y,p,q} XAY^T + XBY^T - p - q$$

subject to

$$AY^T \leq pJ_n^T$$
$$B^T X^T \leq qJ_m^T \qquad (\text{equivalently } XB \leq qJ_m)$$
$$x_i \geq 0, y_j \geq 0, \quad XJ_n = 1 = YJ_m^T$$

where $J_k = (1\ 1\ 1\ \cdots 1)$ is the $1 \times k$ row vector consisting of all 1s. In addition, $p^* = E_I(X^*, Y^*)$, and $q^* = E_{II}(X^*, Y^*)$.

Nonlinear Program (cont'd)

- **Remark.** Expanded, this program reads as

$$\max_{X,Y,p,q} \sum_{i=1}^{n}\sum_{j=1}^{m} x_i a_{ij} y_j + \sum_{i=1}^{n}\sum_{j=1}^{m} x_i b_{ij} y_j - p - q$$

subject to

$$\sum_{j=1}^{m} a_{ij} y_j \leq p, \ i = 1, 2, \ldots, n,$$

$$\sum_{i=1}^{n} x_i b_{ij} \leq q, \ j = 1, 2, \ldots, m,$$

$$x_i \geq 0, y_j \geq 0, \ \sum_{i=1}^{n} x_i = \sum_{j=1}^{m} y_j = 1.$$

   - This is a nonlinear program because of the presence of the terms $x_i y_j$.

Proof of Theorem 3.4.1

- **Part I**: If we have a Nash point, it must solve the nonlinear programming problem.

Proof. Here is how the proof of this useful result goes. Recall that a strategy pair (X^*, Y^*) is a Nash equilibrium if and only if

$$E_{\mathrm{I}}(X^*, Y^*) = X^* A Y^{*T} \geq X A Y^{*T}, \text{ for every } X \in S_n, \qquad (3.4.1)$$

$$E_{\mathrm{II}}(X^*, Y^*) = X^* B Y^{*T} \geq X^* B Y^{T}, \text{ for every } Y \in S_m.$$

Keep in mind that the quantities $E_{\mathrm{I}}(X^*, Y^*)$ and $E_{\mathrm{II}}(X^*, Y^*)$ are scalars. In the first inequality of (3.4.1), successively choose $X = (0, \ldots, 1, \ldots, 0)$ with 1 in each of the n spots, and in the second inequality of (3.4.1) choose $Y = (0, \ldots, 1, \ldots, 0)$ with 1 in each of the m spots, and we see that $E_{\mathrm{I}}(X^*, Y^*) \geq E_{\mathrm{I}}(i, Y^*) = {}_i A Y^{*T}$ for each i, and $E_{\mathrm{II}}(X^*, Y^*) \geq E_{\mathrm{II}}(X^*, j) = X^* B_j$, for each j.

Proof of Theorem 3.4.1 (cont'd)

In matrix form, this is

$$E_{\mathrm{I}}(X^*, Y^*) J_n^T = X^* A Y^{*T} J_n^T \geq A Y^{*T}, \qquad (3.4.2)$$

$$E_{\mathrm{II}}(X^*, Y^*) J_m = (X^* B Y^{*T}) J_m \geq X^* B.$$

However, it is also true that if (3.4.2) holds for a pair (X^*, Y^*) of strategies, then these strategies must be a Nash point, that is, (3.4.1) must be true. Why? Well, if (3.4.2) is true, we choose any $X \in S_n$ and $Y \in S_m$ and multiply

$$E_{\mathrm{I}}(X^*, Y^*) X J_n^T = E_{\mathrm{I}}(X^*, Y^*) = X^* A Y^{*T} X J_n^T \geq X A Y^{*T},$$

$$E_{\mathrm{II}}(X^*, Y^*) J_m Y^T = E_{\mathrm{II}}(X^*, Y^*) = (X^* B Y^{*T}) J_m Y^T \geq X^* B Y^T,$$

because $X J_n^T = J_m Y^T = 1$. But this is exactly what it means to be a Nash point. This means that (X^*, Y^*) is a Nash point if and only if

$$X^* A Y^{*T} J_n^T \geq A Y^{*T}, \quad (X^* B Y^{*T}) J_m \geq X^* B.$$

We have already seen this in Proposition 3.2.3.

Proof of Theorem 3.4.1 (cont'd)

Now suppose that (X^*, Y^*) is a Nash point. We will see that if we choose the scalars

$$p^* = E_\text{I}(X^*, Y^*) = X^* A Y^{*T} \quad \text{and} \quad q^* = E_\text{II}(X^*, Y^*) = X^* B Y^{*T},$$

then (X^*, Y^*, p^*, q^*) is a solution of the nonlinear program. To see this, we first show that all the constraints are satisfied. In fact, by the equivalent characterization of a Nash point we just derived, we get

$$X^* A Y^{*T} J_n^T = p^* J_n^T \geq A Y^{*T} \quad \text{and} \quad (X^* B Y^{*T}) J_m = q^* J_m \geq X^* B.$$

The rest of the constraints are satisfied because $X^* \in S_n$ and $Y^* \in S_m$. In the language of nonlinear programming, we have shown that (X^*, Y^*, p^*, q^*) is a **feasible point**. The **feasible set** is the set of all points that satisfy the constraints in the nonlinear programming problem.

Proof of Theorem 3.4.1 (cont'd)

We have left to show that (X^*, Y^*, p^*, q^*) maximizes the objective function

$$f(X, Y, p, q) = XAY^T + XBY^T - p - q$$

over the set of the possible feasible points.

Since every feasible solution (meaning it maximizes the objective over the feasible set) to the nonlinear programming problem must satisfy the constraints $AY^T \leq pJ_n^T$ and $XB \leq qJ_m$, multiply the first on the left by X and the second on the right by Y^T to get

$$XAY^T \leq pXJ_n^T = p, \qquad XBY^T \leq qJ_mY^T = q.$$

Hence, any **possible** solution gives the objective

$$f(X, Y, p, q) = XAY^T + XBY^T - p - q \leq 0.$$

Proof of Theorem 3.4.1 (cont'd)

So $f(X, Y, p, q) \leq 0$ for any feasible point. But with $p^* = X^* A Y^{*T}, q^* = X^* B Y^{*T}$, we have seen that (X^*, Y^*, p^*, q^*) is a feasible solution of the nonlinear programming problem and

$$f(X^*, Y^*, p^*, q^*) = X^* A Y^* T + X^* B Y^* T - p^* - q^* = 0$$

by definition of p^* and q^*. Hence this point (X^*, Y^*, p^*, q^*) both is feasible and gives the maximum objective (which we know is zero) over any possible feasible solution and so is a solution of the nonlinear programming problem. This shows that if we have a Nash point, it must solve the nonlinear programming problem.

Proof of Theorem 3.4.1 (cont'd)

- **Part II**: Any solution of the nonlinear programming problem must be a Nash point.

Proof.

For the opposite direction, let X_1, Y_1, p_1, q_1 be **any** solution of the nonlinear programming problem, let $(X^*.Y^*)$ be a Nash point for the game, and set $p^* = X^*AY^{*T}$, $q^* = X^*BY^{*T}$. We will show that (X_1, Y_1) must be a Nash equilibrium of the game.

Since X_1, Y_1 satisfy the constraints of the nonlinear program $AY_1^T \leq p_1 J_n^T$ and $X_1 B \leq q_1 J_m$, we get, by multiplying the constraints appropriately

$$X_1 A Y_1^T \leq p_1 X_1 J_n^T = p_1 \quad \text{and} \quad X_1 B Y_1^T \leq q_1 Y_1^T J_m = q_1.$$

Now, we know that if we use the Nash point (X^*, Y^*) and $p^* = X^*AY^{*T}, q^* = X^*BY^{*T}$, then $f(X^*, Y^*, p^*, q^*) = 0$, so zero is the maximum objective. But we have just shown that our **solution** to the program (X_1, Y_1, p_1, q_1) satisfies $f(X_1, Y_1, p_1, q_1) \leq 0$.

Proof of Theorem 3.4.1 (cont'd)

Consequently, it must in fact be equal to zero:

$$f(X_1, Y_1, p_1, q_1) = (X_1 A Y_1^T - p_1) + (X_1 B Y_1^T - q_1) = 0.$$

The terms in parentheses are nonpositive, and the two terms add up to zero. That can happen only if they are each zero. Hence

$$X_1 A Y_1^T = p_1 \quad \text{and} \quad X_1 B Y_1^T = q_1.$$

Then we write the constraints as

$$A Y_1^T \leq (X_1 A Y_1^T) J_n^T, \qquad X_1 B \leq (X_1 B Y_1^T) J_m.$$

However, we have shown at the beginning of this proof that this condition is exactly the same as the condition that (X_1, Y_1) is a Nash point. So that's it; we have shown that any solution of the nonlinear program must give a Nash point, and the scalars must be the expected payoffs using that Nash point. $\square$

Proof of Theorem 3.4.1 (cont'd)

- **Remark.** It is **not** necessarily true that $E_\mathrm{I}(X_1, Y_1) = p_1 = p^* = E_\mathrm{I}(X^*, Y^*)$ and $E_\mathrm{II}(X_1, Y_1) = q_1 = q^* = E_\mathrm{II}(X^*, Y^*)$. Different Nash points can, and usually do, give different expected payoffs, as we have seen many times.

- We restate the Theorem 3.4.1:

  $(X^* \in S_n, Y^* \in S_m)$ *is a Nash equilibrium if and only if they satisfy the*

  nonlinear program

  $$\max_{X, Y, p, q} XAY^T + XBY^T - p - q$$

  subject to

  $$AY^T \le pJ_n^T$$

  $$B^T X^T \le qJ_m^T \qquad (\text{equivalently } XB \le qJ_m)$$

  $$x_i \ge 0, y_j \ge 0, \quad XJ_n = 1 = YJ_m^T$$

  where $J_k = (1\ 1\ 1 \cdots 1)$, $p^* = E_I(X^*, Y^*)$, *and* $q^* = E_{II}(X^*, Y^*)$.

Simple Example

- Consider the matrices

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

  - Before you get started looking for mixed Nash points, you should first find the pure Nash points:

    $(X_1 = (0,1,0), Y_1 = (1,0,0))$ with expected payoff $E_I^1 = 2, E_{II}^1 = 1$

    $(X_2 = (0,0,1) = Y_2)$ with expected payoffs $E_I^2 = 1, E_{II}^2 = 2$.

  - Solve the nonlinear programming and obtain a mixed Nash point:

    $X_3 = \left(0, \frac{2}{3}, \frac{1}{3}\right), \ Y_3 = \left(\frac{1}{3}, 0, \frac{2}{3}\right), \ p = E_I(X_3, Y_3) = \frac{2}{3}, \ q = E_{II}(X_3, Y_3) = \frac{2}{3}.$

Simple Example (cont'd)

- Maple commands to get the solutions:

```
> with(LinearAlgebra):
> A:=Matrix([[-1,0,0],[2,1,0],[0,1,1]]);
> B:=Matrix([[1,2,2],[1,-1,0],[0,1,2]]);
> X:=<x[1],x[2],x[3]>;  #Or X:=Vector(3,symbol=x):
> Y:=<y[1],y[2],y[3]>;  #Or Y:=Vector(3,symbol=y):
> Cnst:={seq((A.Y)[i]<=p,i=1..3),
          seq((Transpose(X).B)[i]<=q,i=1..3),
          add(x[i],i=1..3)=1,add(y[i],i=1..3)=1};
> with(Optimization);
> objective:=expand(Transpose(X).A.Y+Transpose(X).B.Y-p-q);
> QPSolve(objective,Cnst,assume=nonnegative,maximize);
> QPSolve(objective,Cnst,assume=nonnegative,maximize,
                              initialpoint=({q=1,p=2}));
> NLPSolve(objective,Cnst,assume=nonnegative,maximize);
```

Quadratic programming

General nonlinear programming

- There's no need to use a computer to find the pure Nash unless it's a very large game.

Simple Example (cont'd)

- Modify the `assume=nonnegative` commands in case of Nash equilibria associated with negative payoffs.

```
> with(LinearAlgebra):
> A:=Matrix([[-1,0,0],[2,1,0],[0,1,1]]);
> B:=Matrix([[1,2,2],[1,-1,0],[0,1,2]]);
> X:=<x[1],x[2],x[3]>;
> Y:=<y[1],y[2],y[3]>;
> Cnst:={seq((A.Transpose(Y))[i]<=p,i=1..3),seq((X.B)[i]<=q,i=1..3),
            add(x[i],i=1..3)=1,add(y[i],i=1..3)=1,
            seq(y[i]>=0,i=1..3),seq(x[i]>=0,i=1..3)};
> with(Optimization);
> objective:=expand(Transpose(X).A.Y+Transpose(X).B.Y-p-q);
> QPSolve(objective,Cnst,maximize);
> QPSolve(objective,Cnst,maximize,initialpoint=({q=1,p=2}));

> NLPSolve(objective,Cnst,maximize);
```

Example 3.12

- Suppose that two countries are involved in an arms control negotiation. Each country can decide to either cooperate or not cooperate (don't). For this game, one possible bimatrix payoff situation may be

	Cooperate	Don't
Cooperate	(1,1)	(0,3)
Don't	(3,0)	(2,2)

.

   – This game has a pure Nash equilibrium at (2, 2), so these countries will not actually negotiate in good faith. This would lead to what we might call **deadlock** because the two players will decide not to cooperate.

Example 3.12 (cont'd)

- If a third party managed to intervene to change the payoffs, you might get the following payoff matrix:

	Cooperate	Don't
Cooperate	$(3, 3)$	$(-1, -3)$
Don't	$(3, -1)$	$(1, 1)$

  - We now have pure Nash equilibria at both (3,3) and (1,1).
  - Apply the Maple commands, calculus method or formulas, we obtain
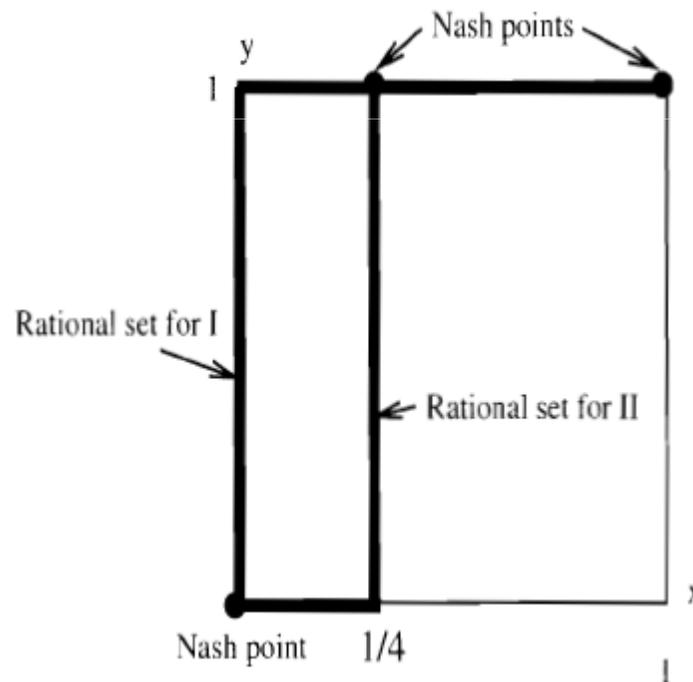
$$X_1 = \left(\frac{1}{4}, \frac{3}{4}\right), Y_1 = (1, 0), p_1 = 3, q_1 = 0,$$
$$X_2 = (0, 1), Y_2 = (0, 1), p_2 = q_2 = 1,$$
$$X_3 = (1, 0) = Y_3, p_3 = q_3 = 3.$$

Example 3.12 (cont'd)

- By graphing the rational reaction sets we see that any mixed strategy
  $X = (x, 1-x), Y = (1,0), \frac{1}{4} \le x \le 1,$ is a Nash point.
- Now each player receives the most if both cooperate.

Example 3.13

- **A Discrete Silent Duel.** Consider a gun duel between two persons, Pierre (player I) and Bill (player II). They each have a gun with exactly one bullet. They face each other initially 10 paces apart. They will walk toward each other. At each step, they each may choose to either fire or hold. If they fire, the probability of a hit depends on how far apart they are according to the following distribution:

$$Prob(\text{Hit}|\text{Paces apart} = k) = \begin{cases} 0.2 & \text{if } k = 10; \\ 0.6 & \text{if } k = 6; \\ 0.8 & \text{if } k = 4; \\ 1 & \text{if } k = 2. \end{cases}$$

Example 3.13 (cont'd)

- First we define the accuracy functions

$$p_1(x) = p_2(x) = \begin{cases} 0.2 & \text{if } x = 0; \\ 0.6 & \text{if } x = 0.4; \\ 0.8 & \text{if } x = 0.6; \\ 1 & \text{if } x = 0.8. \end{cases}$$

Think of $0 \leq x \leq 1$ as the **time** to shoot, $x = 1 - k/10$.

- Define the payoff to player I, Pierre, as

$$u_1(x, y) = \begin{cases} a_1 p_1(x) + b_1(1 - p_1(x))p_2(y) \\ \qquad\qquad + c_1(1 - p_1(x))(1 - p_2(y)) & \text{if } x < y; \\ d_1 p_2(y) + e_1(1 - p_2(y))p_1(x) \\ \qquad\qquad + f_1(1 - p_2(y))(1 - p_1(x)) & \text{if } x > y; \\ g_1 p_1(x)p_2(x) + h_1 p_1(x)(1 - p_2(x)) \\ \quad + k_1(1 - p_1(x))p_2(x) + \ell_1(1 - p_1(x))(1 - p_2(x)) & \text{if } x = y. \end{cases}$$

Example 3.13 (cont'd)

- For example, if $0 \leq x \leq 1$ then Pierre is choosing to fire before Bill and the expected payoff is calculated as

$$u_1(x, y) = a_1 Prob(\text{II killed at } x) + b_1 Prob(\text{I misses at } x) Prob(\text{I killed at } y)$$
$$+ c_1 Prob(\text{I misses at } x) Prob(\text{II misses at } y)$$
$$= a_1 p_1(x) + b_1(1 - p_1(x))(1) + c_1(1 - p_1(x))(1 - p_2(y)).$$

- The **silent** part appears in the case that I misses at *x* and I is killed at *y* because the probability I is killed by II is not necessarily 1 if I misses.

- The constants multiplying the accuracy functions are the payoffs. For Pierre we will use the payoff values

$$a_1 = -2, b_1 = -1, c_1 = 2, d_1 = -1,$$
$$e_1 = 1, f_1 = 2, g_1 = -2, h_1 = 1, k_1 = -1, \ell_1 = 2.$$

Example 3.13 (cont'd)

– The expected payoff to Bill is similarly

$$
u_2(x, y) = \begin{cases}
\begin{aligned}
& a_2 p_1(x) + b_2(1 - p_1(x))p_2(y) \\
& \qquad\qquad + c_2(1 - p_1(x))(1 - p_2(y))
\end{aligned} & \text{if } x < y; \\[2ex]
\begin{aligned}
& d_2 p_2(y) + e_2(1 - p_2(y))p_1(x) \\
& \qquad\qquad + f_2(1 - p_2(y))(1 - p_1(x))
\end{aligned} & \text{if } x > y; \\[2ex]
\begin{aligned}
& g_2 p_1(x)p_2(x) + h_2 p_1(x)(1 - p_2(x)) \\
& \quad + k_2(1 - p_1(x))p_2(x) + \ell_2(1 - p_1(x))(1 - p_2(x))
\end{aligned} & \text{if } x = y.
\end{cases}
$$

– For Bill we will take the payoff values

$$a_2 = -1, b_2 = 1, c_2 = 1, d_2 = 1,$$
$$e_2 = -1, f_2 = 1, g_2 = 0, h_2 = -1, k_2 = 1, \ell_2 = 1.$$

Example 3.13 (cont'd)

— The payoff matrix then for player I is

$$A = (u_1(x, y) : x = 0, 0.4, 0.6, 0.8, y = 0, 0.4, 0.6, 0.8),$$

or

$$A = \begin{bmatrix} 1.20 & -0.24 & -0.72 & -1.2 \\ 0.92 & -0.40 & -1.36 & -1.6 \\ 0.76 & -0.12 & -1.20 & -1.8 \\ 0.6 & -0.2 & -0.6 & -2 \end{bmatrix}.$$

Similarly, player II's matrix is

$$B = \begin{bmatrix} 0.64 & 0.60 & 0.60 & 0.6 \\ 0.04 & 0.16 & -0.20 & -0.2 \\ -0.28 & 0.36 & 0.04 & -0.6 \\ -0.6 & 0.2 & 0.6 & 0 \end{bmatrix}.$$

Example 3.13 (cont'd)

– To solve this game, you may use Maple and adjust the initial point to obtain multiple equilibria. Here is the result:

X	Y	E_{I}	E_{II}
$(0.97, 0, 0, 0.03)$	$(0.17, 0, 0.83, 0)$	-0.4	0.6
$(0.95, 0, 0.03, 0.02)$	$(0.08, 0.8, 0.12)$	-0.19	0.58
$(0.94, 0, 0.06, 0)$	$(0.21, 0.79, 0, 0)$	0.07	0.59
$(0, 0, 0.55, 0.45)$	$(0, 0.88, 0.12, 0)$	-0.25	0.29
$(0, 0, 1, 0)$	$(0, 1, 0, 0)$	-0.12	0.36
$(1, 0, 0, 0)$	$(1, 0, 0, 0)$	1.2	0.64
$(0, 0, 0, 1)$	$(0, 0, 1, 0)$	-0.6	0.6

It looks like the best Nash for each player is to shoot at 10 paces.

Summary of Methods for Finding Mixed Nash Equilibria

① Equality of payoffs. Suppose that we have mixed strategies $X^* = (x_1, \ldots, x_n)$ and $Y^* = (y_1, \ldots, y_m)$. For any rows $k_1, k_2, \ldots$ that have a positive probability of being used, the expected payoffs to player I for using any of those rows must be equal: $E_\mathrm{I}(k_r, Y^*) = E_\mathrm{I}(k_s, Y^*) = E_\mathrm{I}(X^*, Y^*)$. You can find Y^* from these equations. Similarly, for any columns j that have a positive probability of being used, we have $E_\mathrm{II}(X^*, j_r) = E_\mathrm{II}(X^*, j_s) = E_\mathrm{II}(X^*, Y^*)$. You can find X^* from these equations.

② You can use the calculus method directly by computing

$$f(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{m-1}) = \left(x_1, \ldots, x_{n-1}, 1 - \sum_{i=1}^{n-1} x_i \right) A \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m-1} \\ 1 - \sum_{j=1}^{m-1} y_j \end{bmatrix}$$

Summary of Methods for Finding Mixed
Nash Equilibria (cont'd)

and then

$$\frac{\partial f}{\partial x_i} = 0, \quad i = 1, 2, \ldots, n - 1.$$

This will let you find Y^*. Next, compute

$$g(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{m-1}) = \left(x_1, \ldots, x_{n-1}, 1 - \sum_{i=1}^{n-1} x_i \right) B \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ 1 - \sum_{j=1}^{m-1} y_j \end{bmatrix}$$

and then

$$\frac{\partial g}{\partial y_j} = 0, \quad j = 1, 2, \ldots, m - 1.$$

From these you will find X^*.

Summary of Methods for Finding Mixed Nash Equilibria (cont'd)

③ You can use the system of equations to find interior Nash points given by

$$\sum_{j=1}^{m} y_j [a_{kj} - a_{nj}] = 0, \qquad k = 1, 2, \ldots, n-1$$

$$\sum_{i=1}^{n} x_i [b_{is} - b_{im}] = 0, \qquad s = 1, 2, \ldots, m-1.$$

$$x_n = 1 - \sum_{i=1}^{n-1} x_i, \quad y_m = 1 - \sum_{j=1}^{m-1} y_j.$$

④ In the 2×2 case you can find the rational reaction sets for each player and see where they intersect. This gives all the Nash equilibria including the pure ones.

⑤ Use the nonlinear programming method: set up the objective, the constraints, and solve. Use the option initialpoint to modify the starting point the algorithm uses to find additional Nash points.

Choosing Among Several Nash Equilibria ◁

Choosing Among Several Nash Equilibria

- Criteria for choosing among several Nash equilibria
  - Stability (example 3.14)
  - Evolutionary stable strategy (example 3.15)
  - Risk (example 3.16)

Example 3.14

- Let's carry out the repeated best response idea for the two-person zero sum game

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 0 & -2 \\ 0 & -1 & 4 \end{bmatrix}.$$

  - Notice that $v^+ = v^- = 1$ and we have a saddle point at $X^* = (1, 0, 0)$, $Y^* = (0, 1, 0)$.

  - Procedure: We start with any strategy, say, for player II. Then we calculate the best response strategy for player I to this first strategy, then we calculate the best response strategy for player II to the best response for player I, and so on.

Example 3.14 (cont'd)

- Suppose that player II starts by playing column 3. The table summarizes the sequence of best responses:

	Best response Strategy for II	Best response Strategy for I	Payoff to I
Step 0	Column 3 (Start)	Row 3	4
Step 1	Column 2	Row 1	1
Step 2	Column 2	Row 1	1
⋮	⋮	⋮	⋮

- We have arrived at the one and only saddle point of the matrix, namely, I plays row 1 and II plays column 2.

- This convergence to the saddle point will happen no matter where we start with a strategy, and no matter who chooses first.

- This is a really stable saddle point. Because it is the only saddle point?

Example 3.14 (cont'd)

- Here is a matrix with only one saddle but the best response sequence doesn't converge to it:

$$A = \begin{bmatrix} 6 & 3 & 2 \\ 5 & 4 & 5 \\ 2 & 3 & 6 \end{bmatrix}.$$

  - The value is $v(A) = 4,$ and there is a unique saddle at row 2 column 2. Now suppose that the players play as in the following table:

	Best response Strategy for II	Best response Strategy for I
Step 0	Column 1	Row 1
Step 1	Column 3	Row 3
Step 2	Column 1	Row 1
⋮	⋮	⋮

  - Only starting with row 2 or column 2 would bring us to the saddle.

Example 3.15

- Consider the bimatrix game (assume $a > b, c > 0$):

$$\begin{bmatrix} (-a, -a) & (0, -b) \\ (-b, 0) & (-c, -c) \end{bmatrix}.$$

  - We have two pure Nash equilibria at $(-b, 0)$ and at $(0, -b)$.
  - Calculate

$$E_1(x, y) = (x \; 1 - x) \begin{bmatrix} -a & 0 \\ -b & -c \end{bmatrix} \begin{bmatrix} y \\ 1 - y \end{bmatrix} = -ayx - (1 - x)[(b - c)y + c]$$

$$\frac{\partial E_1(x, y)}{\partial x} = -(a - b + c)y + c = 0 \implies y = \frac{c}{a - b + c}.$$

  - Similarly, $x = \dfrac{c}{a - b + c}.$

Example 3.15 (cont'd)

- Defining $h = \dfrac{c}{a - b + c}$, we see that we also have a mixed Nash equilibrium at $X = (h, 1 - h) = Y$.

- Here is the table of payoffs for each of the three equilibria:

	$E_1(x, y)$	$E_2(x, y)$
$x = 0, y = 1$	$-b$	0
$x = 1, y = 0$	0	$-b$
$x = h, y = h$	z	z

where $z = E_1(h, h) = -h^2(a - b + c) - h(b - 2c) - c$.

Example 3.15 (cont'd)

- Suppose $a = 3, b = 1, c = 2$. Then $h = \frac{1}{2}$ gives the mixed Nash equilibrium $X = Y = (\frac{1}{2}, \frac{1}{2})$. These are the payoffs:

Equilibrium	Payoff to I	Payoff to II
$x = y = \frac{1}{2}$	$-\frac{3}{2}$	$-\frac{3}{2}$
$x = 1, y = 0$	0	-1
$x = 0, y = 1$	-1	0

- Without knowing the opponent's choice, they will end up playing $(x = 1, y = 1),$ resulting in the nonoptimal pay off $E_1(1, 1) = -3 = E_2(1, 1)$.

  - If there are **many players playing this game whenever two players encounter each other** and they each play nonoptimally, they will all receive less than they could otherwise get.

Example 3.15 (cont'd)

- If one of the player realizes that, he may decide to switch to play

  $x = 0$ for player I, or $y = 0$ for player II. Then he will receive -1,

  instead of -3.

- But other players would reach this conclusion as well. Consequently others also start playing $x = 0$ or $y = 0$, and now we move again to the nonoptimal play $x = 0, y = 0$, resulting in payoffs -2 to each player.

- If this reasoning is correct, then we could cycle forever between

  $(x = 1, y = 1)$ and $(x = 0, y = 0)$, until someone stumbles on trying

  $x = h = \frac{1}{2}$. Then, $E_1(\frac{1}{2}, 0) = -1$ and $E_2(\frac{1}{2}, 0) = -\frac{3}{2}$.

- Eventually, everyone will see that $\frac{1}{2}$ is a better response to 0 and everyone will switch to $h = \frac{1}{2}$, with payoff $-\frac{3}{2}$.

Example 3.15 (cont'd)

- Notice that since $E_1(x, \frac{1}{2}) = E_2(\frac{1}{2}, y) = -\frac{3}{2}$ for any $x, y,$ **no strategy chosen by either player can get a higher payoff if the opposing player chooses** $h = \frac{1}{2}$.
  - Once a player hits on using $x = \frac{1}{2}$ or $y = \frac{1}{2}$, the cycling is over.

- This Nash equilibrium $x = y = \frac{1}{2}$ is the only one that allows the players to choose without knowing the other's choice and then have no incentive to do something else. It is stable in that sense.

- This strategy is called **uninvadable**, or an **evolutionary stable strategy**, and shows us, sometimes, one way to pick the **right** Nash equilibrium when there are more than one.

Example 3.16

- **Entry Deterrence.** There are two players producing gadgets. Firm (player) A is already producing and selling the gadgets, while firm (player) B is thinking of producing and selling the gadgets and competing with firm A. Firm A has two strategies: (1) join with firm B to control the total market, or

  (2) resist firm B and make it less profitable or unprofitable for firm B to enter the market. Firm B has the two strategies to

  (1) enter the market and compete with firm A or

  (2) move on to something else.

  Here is the bimatrix:

A/B	Enter	Move on
Resist	$(0, -1)$	$(10, 0)$
Join	$(5, 4)$	$(10, 0)$

Example 3.16 (cont'd)

- There are two pure Nash equilibria:

$$X_1 = (1,0), Y_1 = (0,1)$$
$$X_2 = (0,1), Y_2 = (1,0)$$

with associated payoffs

$$(A_1, B_1) = (10,0), \ (A_2, B_2) = (5,4).$$

- Without knowing the opponent's choice, they will play

$X = (1,0), Y = (1,0)$ with the result that A gets 0 and B gets -1.

- In the previous example we did not account for the fact that if there is any positive probability that player B will enter the market, then firm A must take this into account in order to reduce the risk.
  - From this perspective, firm A would definitely not play resist.

Example 3.16 (cont'd)

- Economists say that equilibrium X_2, Y_2 **risk dominates** the other equilibrium and so that is the **correct one.**

- A risk-dominant Nash equilibrium will be **correct** the more uncertainty exists on the part of the players as to which strategy an opponent will choose.

  - The more risk and uncertainty, the more likely the risk-dominant Nash equilibrium will be played.

Pareto-Optimal and Payoff-Dominant

- **Definition 3.5.1** *Given a collection of payoff functions*

$$(u_1(q_1, \ldots, q_n), \ldots, u_n(q_1, \ldots, q_n))$$

for an n–person nonzero sum game, where the q_i is a pure or mixed strategy for player $i = 1, 2, \ldots, n$, we say that $(q_1^*, \ldots, q_n^*)$ is Pareto-optimal if there does not exist any other strategy for any of the players that makes that player better off, that is, increases her or his payoff, without making other players worse off, namely, decreasing at least one other player's payoff.

  - In the Entry Deterrence example, $(5, 4)$ is the Pareto-optimal payoff point. If either player deviates from using $X_2, Y_2,$ then at least one of the two players does worse.

-

Pareto-Optimal and Payoff-Dominant (Cont'd)

– On the other hand, if we look back at the prisoner's dilemma problem at the beginning of this chapter we showed that (-5,-5) is a Nash equilibrium, but it is not Pareto-optimal because (-1,-1) simultaneously improves both their payoffs.

Definition 3.5.2 *A Nash equilibrium is* **payoff-dominant** *if it is Pareto-optimal compared to all other Nash equilibria in the game.*

Payoff-Dominant and Risk-Dominant

- Here is an example, commonly known as the **stag hunt game:**

	Hunt	Gather
Hunt	$(5,5)$	$(0,4)$
Gather	$(4,0)$	$(2,2)$

- This is an example of a **coordination** game. If the players can coordinate their actions and hunt, then they can both do better.
- The following table summarizes the Nash points and their payoffs:

$X_1 = (\frac{2}{3}, \frac{1}{3}) = Y_1$	$E_{\mathrm{I}} = \frac{10}{3}$	$E_{\mathrm{II}} = \frac{10}{3}$
$X_2 = (0,1) = Y_2$	$E_{\mathrm{I}} = 2$	$E_{\mathrm{II}} = 2$
$X_3 = (1,0) = Y_3$	$E_{\mathrm{I}} = 5$	$E_{\mathrm{II}} = 5$

Payoff-Dominant and Risk-Dominant (cont'd)

- ## Payoff-dominant
  - The Nash equilibrium X_3, Y_3 is payoff-dominant because no player can do better no matter what.

- ## Risk-dominant
  - The Nash equilibrium X_2, Y_2 risk dominates X_3, Y_3; i.e., (gather,gather) risk dominates (hunt,hunt).
  - The intuitive reasoning is that if either player is not absolutely certain that the other player will join the hunt, then the player who was going to hunt sees that she can do better by gathering.
  - Both players play gather in order to minimize the risk of getting zero.