Game Theory

Chapter 1

Matrix Two-Person Games

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Contents

The Basic

- **A game** involves a number of players N, a set of strategies for each player, and a payoff that quantitatively describes the outcome of each play of the game in terms of the amount that each player wins or loses.

- **A strategy** for each player can be very complicated because it is a plan, determined at the start of the game, that describes what a player will do in every possible situation.

A Two-Person Zero Sum Game

- Illustration (two players)
  - $\begin{cases} \text{Player I with n possible strategies (strategy i, i = 1,…, n)} \\ \text{Player II with m possible strategies (strategy j, j = 1,…, m)} \end{cases}$
  - Payoff (game) matrix

	player II			
player I	Strategy 1	Strategy 2	$\cdots$	Strategy m
Strategy 1	a_{11}	a_{12}	$\cdots$	a_{1m}
Strategy 2	a_{21}	a_{22}		a_{2m}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Strategy n	a_{n1}	a_{n2}	$\cdots$	a_{nm}

    - a_{ij} : The payoff to player I
  - **Zero sum games:** Whatever one player wins the other player loses.
    - If the payoff to player I is a_{ij}, then the payoff to player II is $-a_{ij}$.
  - **Both players want to choose strategies that will maximize their individual playoffs.**

Constant Sum Games

- Constant sum games
  - **If the payoff to player I is a_{ij} , then the payoff to player II is $C - a_{ij}$, where C is a fixed constant.**
  - In a zero sum game $C = 0$.
  - The optimal strategies for each player will not change even if we think of the game as zero sum.
    - If we solve it as if the game were zero sum to get the optimal result for player I, then the optimal result for player II would be simply C minus the optimal result for I.

Example 1.1

- Pitching in baseball
  - Let player I be the batter and player II be the pitcher. The pitcher can throw a fastball (F), curve (C), or slider (S), and the batter can expect one of these three pitches and to prepare for it.
  - Payoff matrix

I/II	F	C	S
F	0.30	0.25	0.20
C	0.26	0.33	0.28
S	0.28	0.30	0.33

    - For example, if the batter looks for a fastball and the pitcher actually pitches a fastball, then player I has probability 0.30 of getting a hit.
  - This is a constant sum game because player II's payoff and player I's payoff actually add up to 1.

Example 1.2

- Two companies are both thinking about introducing competing products into the marketplace. They choose the time to introduce the product, and their choices are 1 month, 2months, or 3 months.
  - Payoff matrix

I/II	1	2	3
1	0.5	0.6	0.8
2	0.4	0.5	0.9
3	0.2	0.7	0.5

    - For instance, if player I introduces the product in 3 months and player II introduces it in 2months, the it will turn out that player I will get 70% of the market.
  - The companies want to introduce the product in order to maximize their market share.
  - A constant sum game.

Example 1.3

- An evader (called Rat) is forced to run a maze entering at point A, and a pursuer (called Cat) will also enter the maze at point B. Rat and Cat will run exactly four segments of the maze and the game ends.
  - If Cat and Rat ever meet at an intersection point of segments at the same time, Cat wins +1 and Rat loses -1, while if they never meet during the run, both Cat and Rat win 0.

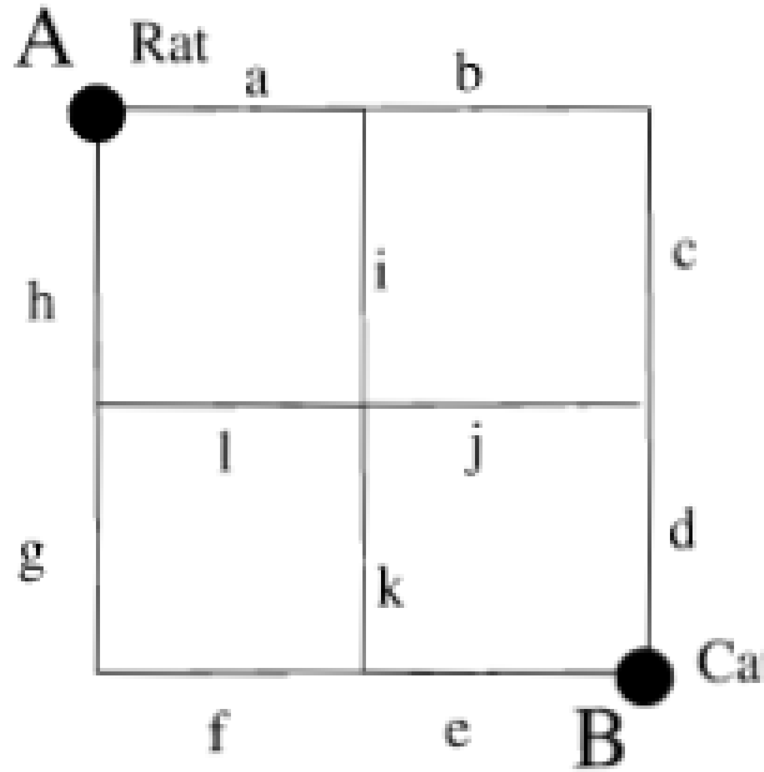


Figure 1.1 Maze for Cat vs. Rat

Example 1.3 (cont'd)

- A zero sum game.
- Cat wants to maximize the payoffs, while Rat wants to minimize them.
- With four segments the payoff matrix will turn out to be a 16×16 one.

Cat/Rat	abcd	abcj	⋯	hlkf
dcba	1	1	⋯	0
dcbi	1	1	⋯	0
⋮	⋮	⋮	⋮	⋮
eklg	0	0	⋯	1

Example 1.4

- 2×2 Nim.
  - Four pennies are set out in two piles of two pennies each.
  - Player I chooses a pile and then decides to remove one or two pennies from the pile chosen.
  - Then player II chooses a pile with at least one penny and decides how many pennies to remove.
  - Then player I starts the second round with the same rules.
  - When both piles have no pennies, the game ends and the loser is the player who removed the last penny.
  - The loser pays the winner one dollar.

Example 1.4 (cont'd)

– Extensive form: represents a game by a tree

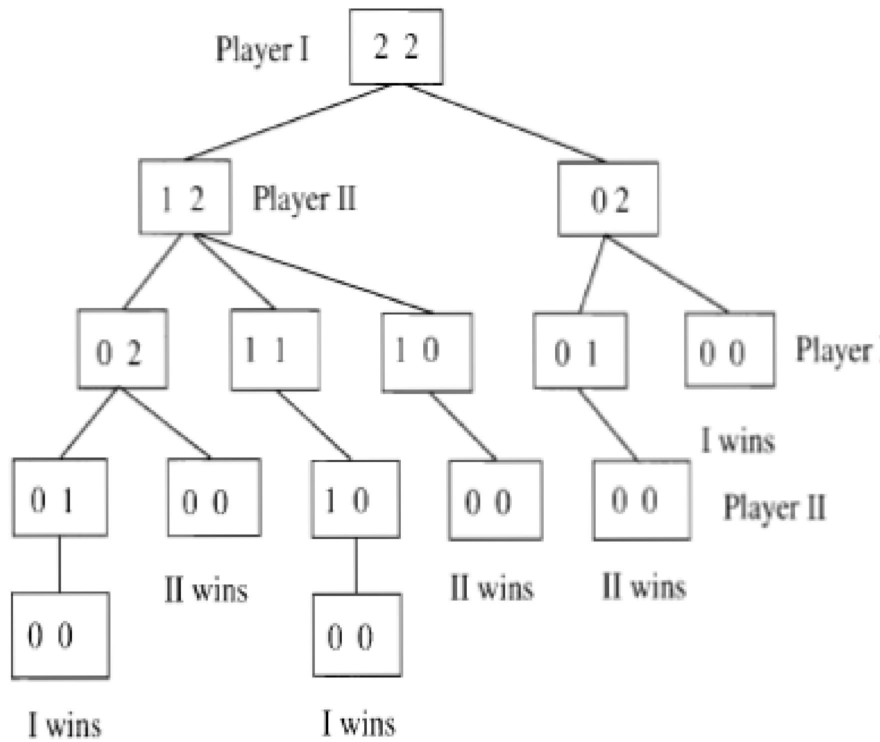


Figure 1.2 2 × 2 Nim tree

Example 1.4 (cont'd)

– Strategies

Strategies for player I
(1) Play (1,2) then, if at (0,2) play (0,1).
(2) Play (1,2) then if at (0,2) play (0,0).
(3) Play (0,2).

Strategies for player II
(1) If at (1,2) → (0,2); if at (0,2) → (0,1)
(2) If at (1,2)→ (1,1); if at (0,2)→ (0,1)
(3) If at (1,2)→ (1,0); if at (0,2)→ (0,1)
(4) If at (1,2)→ (0,2); if at (0,2)→ (0,0)
(5) If at (1,2)→ (1,1); if at (0,2)→ (0,0)
(6) If at (1,2)→ (1,0); if at (0,2)→ (0,0)

– Payoff matrix

player I/player II	1	2	3	4	5	6
1	1	1	−1	1	1	−1
2	−1	1	−1	−1	1	−1
3	−1	−1	−1	1	1	1

Example 1.4 (cont'd)

- Analysis of 2×2 Nim.
  - Any rational player in II's position would drop column 5 In the payoff matrix from consideration (column 5 is called a **dominated strategy**).

    By the same token, player I would drop column 3 from consideration.
  - The **value of this game** is -1 and the strategies $(\mathrm{I}1, \mathrm{II}3), (\mathrm{I}2, \mathrm{II}3), (\mathrm{I}3, \mathrm{II}3)$ are **saddle points**, or **optimal strategies** for the players.
    - **Player I can improve the payoff if player II deviates from column 3.**
    - There are three saddle points in this example, so saddles are not necessarily unique.

Example 1.5

- Russian roulette
  - Two players are faced with a six-shot pistol loaded with one bullet.
  - The players ante $1000, and player I goes first.
  - At each play of the game, a player has the option of putting an additional $1000 into the pot and passing, or spinning the chamber and firing (at his own head).

Example 1.5 (cont'd)

– Game tree
  - The numbers at the end of the branches are the payoffs to player I.
  - The circled nodes are spots at which the next node is decides by chance.
  - The dotted line indicates the optimal strategies.

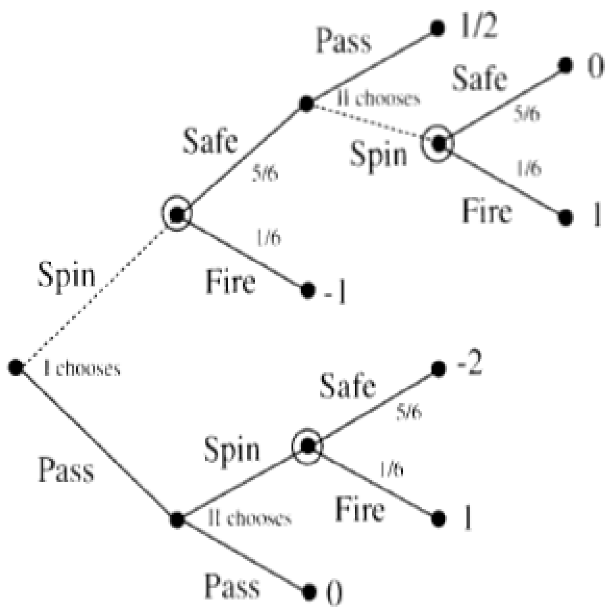


Figure 1.3 Russian roulette

Example 1.5 (cont'd)

– Strategies

Player I

I1	S
I2	P

Player II

II1	If I2, then S; If I1, then P.
II2	If I2, then P; If I1, then P.
II3	If I1, then S; If I2, then P.
II4	If I1, then S; If I2, then S.

– Payoff matrix

I/II	II1	II2	II3	II4
I1	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{36}$	$-\frac{1}{36}$
I2	$-\frac{3}{2}$	0	0	$-\frac{3}{2}$

- Computation

$$\text{I1 against II1}: \frac{5}{6}\left(\frac{1}{2}\right) + \frac{1}{6}(-1) = \frac{1}{4},$$

$$\text{I2 against II1}: \frac{5}{6}(-2) + \frac{1}{6}(1) = -\frac{3}{2}.$$

$$\text{I1 against II3}: \frac{5}{6}\left(\frac{5}{6}(0) + \frac{1}{6}(1)\right) + \frac{1}{6}(-1) = -\frac{1}{36},$$

$$\text{I2 against II4}: \frac{5}{6}(-2) + \frac{1}{6}(1) = -\frac{3}{2}.$$

Example 1.6

- Evens or Odds.
  - Each player decides to show one, two, or three fingers. If the total number of fingers shown is even, player I wins +1 and player II loses -1. If the total number of fingers shown is odd, player I loses -1 and player II wins +1.
  - Payoff matrix

Evens	Odds		
I/II	1	2	3
1	1	−1	1
2	−1	1	−1
3	1	−1	1

  - The row player here will always want to maximize his payoff, while the column player wants to minimize the payoff to the row player.
  - The rows are called the pure strategies for player I, and the columns are called the pure strategies for player II.

Example 1.6 (cont'd)

- If a player always plays the same strategy, the opposing player can win the game.
- It seems that the only alternatives is for the players to **mix up their strategies** and play some rows and columns sometimes and other rows and columns at other times.

Def. 1.1 (cont'd)

- Analysis
  - Player I assumes that player II is playing her best, so II chooses a column *j* so as to

$$Minimize \quad a_{ij} \quad over \quad j = 1, \ldots, m$$

for any given row *i.* Then player I can guarantee that he can choose the row *i* that will maximize this. So player I can **guarantee** that in the **worst possible situation** he can get at least

$$v^- \equiv \max_{i=1,\ldots,n} \min_{j=1,\ldots,m} a_{ij},$$

and we call v^- the **lower value of the game**, which represents the least amount that player I can be guaranteed to receive.

Def. 1.1 (cont'd)

– Player II assumes that player I is playing his best, so I chooses a row *i* so as to

$$Maximize \quad a_{ij} \quad over \quad i = i,\ldots,n$$

for any given column *j*. Player II can therefore choose her column *j* so as to **guarantee** a loss of no more than

$$v^{+} \equiv \min_{j=1,\ldots,m} \max_{i=1,\ldots,n} a_{ij}$$

and we call v^{+} the **upper value of the game**, which represents the largest amount that player II can guarantee can be lost.

– Clearly, $v^{-} \leq v^{+}$

Def. 1.1 (cont'd)

- Find the upper and lower value for any given matrix
  - Game matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

  - For each row, find the minimum payoff in each column and write it in a new additional last column. Then the lower value is the largest number in that last column.
  - In each column find the maximum of the payoffs (written in the last row). The upper value is the smallest of those numbers in the last row.

$$\begin{array}{cccc|l} a_{11} & a_{12} & \cdots & a_{1m} & \longrightarrow \min_j a_{1j} \\ a_{21} & a_{22} & \cdots & a_{2m} & \longrightarrow \min_j a_{2j} \\ \vdots & \vdots & \cdots & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nm} & \longrightarrow \min_j a_{nj} \\ \hline \downarrow & \downarrow & \cdots & \downarrow & \\ \max_i a_{i1} & \max_i a_{i2} & \cdots & \max_i a_{im} & \quad v^- = \text{largest min} \\ & & & & \quad v^+ = \text{smallest max} \end{array}$$

Def. 1.1 (cont'd)

- **Definition 1.1.1** *A matrix game with matrix* $A_{n\times m} = (a_{ij})$ *has the lower value* $v^- \equiv \max_{i=1,\dots,n} \min_{j=1,\dots,m} a_{ij}$

  and the upper value $v^+ \equiv \min_{j=1,\dots,m} \max_{i=1,\dots,n} a_{ij}$.

  – v^- is player I's gain floor, and v^+ is player II's loss ceiling.
  – The **game has a value** if $v^- = v^+$, and we write it as $v = v(A) = v^+ = v^-$. This means that the smallest max and the largest min must be equal and the row and column i^*, j^* giving the payoffs $a_{i^*,j^*} = v^+ = v^-$ are **optimal,** or a **saddle point in pure strategies.**
  – **A fair game**
    - If the value *v* is positive, player I should pay player II the amount *v.* If $v < 0$, then player II should pay player I the amount $-v$.

Example 1.7

- 2×2 Nim.

1	1	−1	1	1	−1	$\longrightarrow$	min = −1
−1	1	−1	−1	1	−1	$\longrightarrow$	min = −1
−1	−1	−1	1	1	1	$\longrightarrow$	min = −1
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$		$v^- = -1$
max = 1	max = 1	max = −1	max = 1	max = 1	max = 1	$v^+ = -1$	

  - $v^+ = v^- = -1$ and so 2×2 Nim has a value $v = -1$.
  - The optimal strategies are located as the (row,column) where the smallest max is -1 and the largest min is also -1.
    - This occurs at any row for player I, but player II must play column 3, so $i^* = 1,2,3 \quad j^* = 3$.
    - The optimal strategies are **not** at **any** row column combination giving -1 as the payoff.

Example 1.7 (cont'd)

- $v^- \le v^+$ verification : The most that I can be guaranteed to win should be less than (or equal to) the most that II can be guaranteed to lose)

- For any fixed row i, $\min_j a_{ij} \le a_{ij}$

$$\Rightarrow v^- = \max_i \min_j a_{ij} \le \max_i a_{ij}$$

$$\Rightarrow v^- = \max_i \min_j a_{ij} \le \min_j \max_i a_{ij} = v^+$$

Def. 1.1.2 and Lemma 1.1.3

- **Definition 1.1.2** *We call a particular row i* and column j* a* **saddle point in pure strategies** *of the game if*

$$a_{ij^*} \le a_{i^*j^*} \le a_{i^*j}, \ \textit{for all rows } i = 1, \ldots, n \textit{ and columns } j = 1, \ldots, m. \tag{1.1.1}$$

- **Lemma 1.1.3** A *game will have a saddle point in pure strategies if and only if*

$$v^- = \max_i \min_j a_{ij} = \min_j \max_i a_{ij} = v^+. \tag{1.1.2}$$

Proof of Lemma 1.1.3 (cont'd)

- Prove lemma 1.1.3
  - If (1.1.1) is true, then

$$v^+ = \min_j \max_i a_{ij} \le \max_i a_{i,j^*} \le a_{i^*,j^*} \le \min_j a_{i^*,j} \le \max_i \min_j a_{i,j} = v^-.$$

    That is, $v^+ \le v^-$. But $v^- \le v^+$ always, so $v = v^+ = v^- = a_{i^*,j^*}$.
  - If $v^+ = v^-$ then

$$\min_j \max_i a_{i,j} = \max_i \min_j a_{i,j}$$

    Let j^* be such that $v^+ = \max_i a_{i,j^*}$ and i^* such that $v^- = \min_j a_{i^*,j}$.

    Then,

$$a_{i^*,j} \ge v^- = v^+ \ge a_{i,j^*}, \quad for \quad any \quad i = 1,\ldots,n, \quad j = 1,\ldots,m.$$

    In addition, taking $j = j^*$ on the left, and $i = i^*$ on the right, gives

$$a_{i^*,j^*} = v^+ = v^-.$$

□

Best Reponse (cont’d)

- When a saddle point exists in pure strategies, (1.1.1) says that if any player deviates from playing her part of the saddle, then the other player can take advantage and improve his payoff.

- Each part of a saddle is a **best response** to the other.

Example 1.8

- In the baseball example player I, the batter, expects the pitcher (player II) to throw a fastball, a slider, or a curveball.
  - Game matrix

I/II	F	C	S
F	0.30	0.25	0.20
C	0.26	0.33	0.28
S	0.28	0.30	0.33

  - A quick calculation shows that $v^- = 0.28$ and $v^+ = 0.30$. So baseball does not have a saddle point in pure strategies.
  - That shouldn't be a surprise because if there were such a saddle, baseball would be a very dull game.

Find the Upper and Lower Values With Maple

- Information

$$A = \begin{bmatrix} 2 & -5 \\ -3 & 1 \\ 4 & -3 \end{bmatrix}, \quad v^- = -3, \quad v^+ = 1.$$

- Maple commands

```
> with(LinearAlgebra):
> A:=Matrix([[2,-5],[-3,1],[4,-3]]);
> rows:=3 : cols:=2:
> vu:=min(seq(max(seq(A[i,j],i=1..rows)),j=1..cols));
> vl:=max(seq(min(seq(A[i,j],j=1..cols)),i=1..rows));
> print("the upper value is",vu);
> print("the lower value is",vl);
```

  - The number of rows and columns could be obtained from Maple by using the statement

```
rows:=RowDimension(A); cols:=ColumnDimension(A).
```

The Von Neumann Minimax Theorem

Mixed Strategies

- What do we do when $v^- < v^+$? If optimal pure strategies don't exist, then how do we play the game ?
-
- John von Neumann figured out how to model **mixing** strategies in a game mathematically and then proved that if we allow **mixed strategies** in a matrix game, it will always have a value and optimal strategies.

Def. 1.2.1

- **Definition 1.2.1** *Let C and D be sets. A function* $f : C \times D \to R$ *has at least one saddle point* (x*, *y*) with* $x^* \in C$ *and* $y^* \in D$ *if*

$$f(x, y^*) \le f(x^*, y^*) \le f(x^*, y) \quad for \quad all \quad x \in C, y \in D.$$

  - Once again we could define the upper and lower values for the game defined using the function *f*, called a **continuous game,** by

$$v^+ = \min_{y \in D} \max_{x \in C} f(x, y), \quad and \quad v^- = \max_{x \in C} \min_{y \in D} f(x, y).$$

  - Check as before that $v^- \le v^+$. If it turns out that $v^+ = v^-$ we say, as usual, that the **game has a value** $v = v^+ = v^-$.

Def. 1.2.2 (cont'd)

- **Definition 1.2.2** *A set* $C \subset R^n$ *is* **convex** *if for any two points* $a, b \in C$ *and all scalars* $\lambda \in [0,1]$, *the line segment connecting* a *and b is also in C, i.e., for all* $a, b \in C,\ \lambda a + (1-\lambda) b \in C,$ $\forall 0 \le \lambda \le 1.$
  - *C is closed if it contains all limit points of sequences in C;*
  - *C is bounded if it can be jammed inside a ball for some large enough radius.*
  - *A closed and bounded subset of Euclidean space is compact.*

Def. 1.2.2 (cont'd)

- For any $a,b \in C, 0 \le \lambda \le 1$,
  a function $g : C \to R$ *is* ***convex*** *if*

$$g(\lambda a + (1-\lambda)b) \le \lambda g(a) + (1-\lambda)g(b);$$

  This says that the line connecting $g(a)$ *with* $g(b)$, *namely* $\{\lambda g(a) + (1-\lambda)g(b) : 0 \le \lambda \le 1\}$, *must always lie above the function values* $g(\lambda a + (1-\lambda)b)$, $0 \le \lambda \le 1$.

  a function $g : C \to R$ *is* ***concave*** *if*

$$g(\lambda a + (1-\lambda)b) \ge \lambda g(a) + (1-\lambda)g(b).$$

- *A function is strictly convex or concave, if the inequalities are strict.*

Def. 1.2.2 (cont'd)

– Figure 1 .4 compares a convex set and a nonconvex set. Also, recall the common calculus test for twice differentiable functions of one variable. If $g = g(x)$ is a function of one variable and has at least two derivatives, then g is convex if $g'' > 0$ and g is concave if $g'' < 0$.

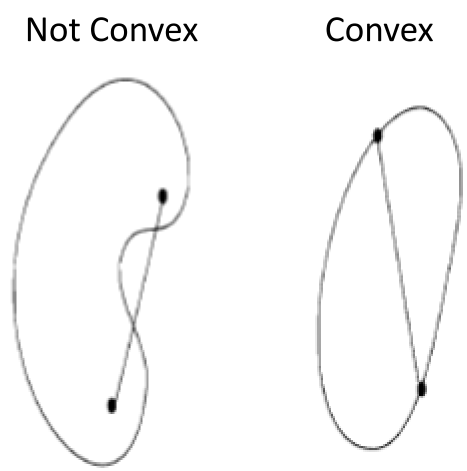


Figure 1.4 Convex and nonconvex sets

The Von Neumann Minimax Theorem (cont'd)

- **Theorem 1.2.3** *Let* $f : C \times D \to R$ *be a continuous function. Let* $C \in R^n$ *and* $D \in R^m$ *be convex, closed, and bounded. Suppose that* $x \mapsto f(x, y)$ *is concave and* $y \mapsto f(x, y)$ *is convex. Then*

$$v^+ = \min_{y \in D} \max_{x \in C} f(x, y) = \max_{x \in C} \min_{y \in D} f(x, y) = v^-$$

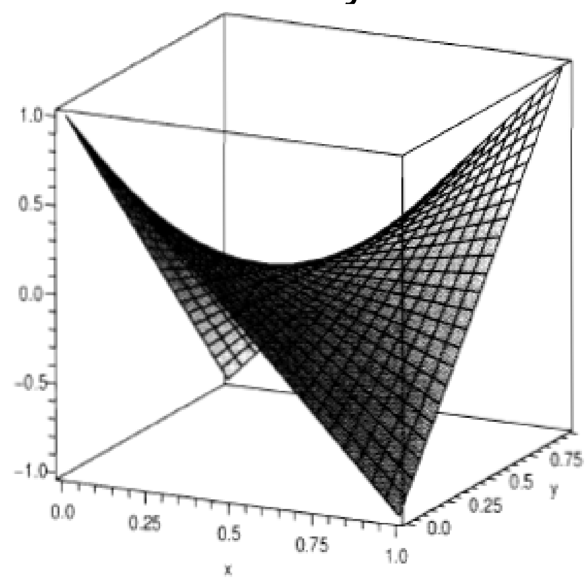
  - For example, $f(x, y) = 4xy - 2x - 2y + 1 \quad on \quad 0 \le x, y \le 1.$
    This function has $f_{xx} = 0 \ge 0,\ f_{yy} = 0 \le 0,$ so it is convex in *y* for each *x* and concave in *x* for each *y.*
  - Since $(x, y) \in [0,1] \times [0,1]$, and the square is closed and bounded, von Neumann's theorem guarantees the existence of a saddle point for this function.

The Von Neumann Minimax Theorem (cont'd)

– Solve $f_x = f_y = 0$ to get $x = y = \frac{1}{2}$. The Hessian for *f*, which is the matrix of second partial derivatives, is given by

$$H(f,[x,y]) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}.$$

Since det(*H*) = -16 < 0 we are guaranteed by elementary calculus that $(x = y = \frac{1}{2})$ is an interior saddle for *f*. Here is a Maple generated picture of *f*:



The Von Neumann Minimax Theorem (cont'd)

– Another way to write our example function would be

$$f(x,y) = (x, 1-x)A(y, 1-y)^T, \text{ where } A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

We will see that $f(x,y)$ is constructed from a matrix game in which player I uses the variable **mixed strategy** $X = (x,1-x)$, and player II uses the variable **mixed strategy** $Y = (y,1-y)$.

– Obviously, not all functions will have saddle points. For instance, $g(x,y) = (x - y)^2$ is not concave-convex and in fact does not have a saddle point in [0,1] x [0,1].

Prove the Von Neumann Minimax Theorem

- **Proof 1.** Define the sets of points where the min or max is attained by

$$B_x := \{y^0 \in D : f(x, y^0) = \min_{y \in D} f(x, y)\} \quad \text{for each fixed } x \in C,$$

$$A_y := \{x^0 \in C : f(x^0, y) = \max_{x \in C} f(x, y)\} \quad \text{for each fixed } y \in D.$$

  – By the assumptions on *f*, C, *D,* these sets are nonempty, closed, and convex. For instance, here is why B_x is convex. Take $y_1^0, y_2^0 \in B_x$, and let $\lambda \in (0,1)$. Then

$$f(x, \lambda y_1^0 + (1-\lambda) y_2^0) \le \lambda f(x, y_1^0) + (1-\lambda) f(x, y_2^0) = \min_{y \in D} f(x, y).$$

  But $f(x, \lambda y_1^0 + (1-\lambda) y_2^0) \ge \min_{y \in D} f(x, y)$ as well, and so they must be equal. This means that $\lambda y_1^0 + (1-\lambda) y_2^0 \in B_x$.

Prove the Von Neumann Minimax Theorem (cont'd)

- Define $g(x, y) \equiv A_y \times B_x$, which takes a point $(x, y) \in C \times D$ and gives the set $A_y \times B_x$, This function satisfies the continuity properties required by Kakutani's theorem. Furthermore, $A_y \times B_x$ are nonempty, convex, and closed, and so Kakutani's theorem says that there is a point $(x^*, y^*) \in g(x^*, y^*) = A_{y^*} \times B_{x^*}$. Writing out what this says, we get

$$f(x^*, y^*) = \max_{x \in C} f(x, y^*) \quad \text{and} \quad f(x^*, y^*) = \min_{y \in D} f(x^*, y),$$

  so that

$$v^+ = \min_{y \in D} \max_{x \in C} f(x, y) \le f(x^*, y^*) \le \max_{x \in C} \min_{y \in D} f(x, y) = v^- \le v^+$$

  and $$f(x, y^*) \le f(x^*, y^*) \le f(x^*, y), \ \forall\, x \in C, y \in D.$$

- This says that (x^*, y^*) is a saddle point and $v = v^+ = v^- = f(x^*, y^*)$.

□

Kakutani's Theorem

- **Theorem 1.2.4** *Let C be a closed, bounded, and convex subset of* R^n*, and let g be a point (in C) to set (subsets of C) function. Assume that for each* $x \in C,$ *the set g(x) is nonempty and convex. Also assume that g is (upper semi)*[5] *continuous. Then there is a point* $x^* \in C$ *satisfying* $x^* \in g(x^*)$.
  - Kakutani's theorem is a **fixed-point theorem.**
  - A fixed-point theorem gives conditions under which a function has a point x* that satisfies *f(x*) = x*,* so *f* fixes the point *x*.*
  - Later use Kakutani's theorem to show that a generalized saddle point, called **Nash equilibrium,** is a fixed point.
  - [5] That is, for any sequences $x_n \in C, y_n \in g(x_n)$, if $x_n \to x$ and $y_n \to y$, then $y \in g(x)$.

Prove the Von Neumann Minimax Theorem (cont'd)

- **Proof 2.**
  1. Assume first that *f* is *strictly* concave-convex, meaning that

$$f(\lambda x + (1-\lambda)z, y) > \lambda f(x,y) + (1-\lambda) f(z,y),\ 0 < \lambda < 1,$$
$$f(x, \mu y + (1-\mu)w) < \mu f(x,y) + (1-\mu) f(x,w),\ 0 < \mu < 1.$$

  - The advantage of doing this is that for each $x \in C$ there is one and only one $y = y(x) \in D$ *(y* depends on the choice of *x)* so that

$$f(x, y(x)) = \min_{y \in D} f(x,y) := g(x).$$

  - This defines a function $g: C \to R$ that is continuous (since *f* is continuous on the closed bounded sets *C x D* and thus is continuous).

Prove the Von Neumann Minimax Theorem (cont'd)

- *g(x)* is concave since

$$g(\lambda x + (1-\lambda)z) \geq \min_{y \in D}(\lambda f(x,y) + (1-\lambda) f(z,y)) \geq \lambda g(x) + (1-\lambda) g(z).$$

- So, there is a point $x^* \in C$ at which *g* achieves its maximum:

$$g(x^*) = f(x^*, y(x^*)) = \max_{x \in C} \min_{y \in D} f(x,y).$$

Prove the Von Neumann Minimax Theorem (cont'd)

- 2. Let $x \in C$ and $y \in D$ be arbitrary. Then, for any $0 < \lambda < 1$, we obtain

$$\begin{aligned} f(\lambda x + (1-\lambda)x^*, y) &> \lambda f(x, y) + (1-\lambda) f(x^*, y) \\ &\geq \lambda f(x, y) + (1-\lambda) f(x^*, y(x^*)) \\ &= \lambda f(x, y) + (1-\lambda) g(x^*). \end{aligned}$$

Take $y = y(\lambda x + (1-\lambda)x^*) \in D$ to get

$$\begin{aligned} g(x^*) &\geq f(\lambda x + (1-\lambda)x^*, y(\lambda x + (1-\lambda)x^*)) = g(\lambda x + (1-\lambda)x^*) \\ &\geq g(x^*)(1-\lambda) + \lambda f(x, y(\lambda x + (1-\lambda)x^*)), \end{aligned}$$

where the first inequality follows from the fact that

$g(x^*) \geq g(x), \quad \forall x \in C.$

Prove the Von Neumann Minimax Theorem (cont'd)

As a result, we have,

$$g(x^*)[1-(1-\lambda)] = g(x^*)\lambda \geq \lambda f(x, y(\lambda x + (1-\lambda)x^*)),$$

or

$$f(x^*, y(x^*)) = g(x^*) \geq f(x, y(\lambda x + (1-\lambda)x^*)) \text{ for all } x \in C.$$

Prove the Von Neumann Minimax Theorem (cont'd)

- 3. Sending $\lambda \to 0$, we see that $\lambda x + (1-\lambda)x^* \to x^*$ and $y(\lambda x + (1-\lambda)x^*) \to y(x^*)$. We obtain

$$f(x, y(x^*)) \le f(x^*, y(x^*)) := v, \text{ for any } x \in C.$$

Consequently, with $y^* = y(x^*)$

$$f(x, y^*) \le f(x^*, y^*) = v, \quad \forall\, x \in C.$$

In addition, since $f(x^*, y^*) = \min_y f(x^*, y) \le f(x^*, y) \quad for \quad all \quad y \in D$, we get

$$f(x, y^*) \le f(x^*, y^*) = v \le f(x^*, y), \forall\, x \in C, y \in D.$$

Prove the Von Neumann Minimax Theorem (cont'd)

This says that (x^*, y^*) is a saddle point and the minimax theorem holds, since

$$\min_y \max_x f(x,y) \le \max_x f(x, y^*) \le v \le \min_y f(x^*, y) \le \max_x \min_y f(x,y),$$

and so we have equality throughout because the right side is always less than the left side.

Prove the Von Neumann Minimax Theorem (cont'd)

- 4. The last step would be to get rid of the assumption of strict concavity and convexity. For $\varepsilon > 0$ set

$$f_\varepsilon(x,y) \equiv f(x,y) - \varepsilon|x|^2 + \varepsilon|y|^2, \quad |x|^2 = \sum_{i=1}^{n} x_i^2, \quad |y|^2 = \sum_{j=1}^{m} y_j^2.$$

  This function will be strictly concave-convex, so the previous steps apply to f_ε. Therefore, we get a point $(x_\varepsilon, y_\varepsilon) \in C \times D$ so that $v_\varepsilon = f_\varepsilon(x_\varepsilon, y_\varepsilon)$ and

$$f_\varepsilon(x, y_\varepsilon) \le v_\varepsilon = f_\varepsilon(x_\varepsilon, y_\varepsilon) \le f_\varepsilon(x_\varepsilon, y), \quad \forall\, x \in C, y \in D.$$

  Since $f_\varepsilon(x, y_\varepsilon) \ge f(x, y_\varepsilon) - \varepsilon|x|^2$ and $f_\varepsilon(x_\varepsilon, y) \le f(x_\varepsilon, y) + \varepsilon|y|^2$, we get

$$f(x, y_\varepsilon) - \varepsilon|x|^2 \le v_\varepsilon \le f(x_\varepsilon, y) + \varepsilon|y|^2, \quad \forall\,(x,y) \in C \times D.$$

Prove the Von Neumann Minimax Theorem (cont'd)

Since $f_\varepsilon(x, y_\varepsilon) \geq f(x, y_\varepsilon) - \varepsilon|x|^2$ and $f_\varepsilon(x_\varepsilon, y) \leq f(x_\varepsilon, y) + \varepsilon|y|^2$, we get

$$f(x, y_\varepsilon) - \varepsilon|x|^2 \leq v_\varepsilon \leq f(x_\varepsilon, y) + \varepsilon|y|^2, \ \ \forall\, (x, y) \in C \times D.$$

Since the sets *C, D* are closed and bounded, we take a sequence $\varepsilon \to 0, x_\varepsilon \to x^* \in C, y_\varepsilon \to y^* \in D$ and also $v_\varepsilon \to v \in R$. Sending $\varepsilon \to 0$, we get

$$f(x, y^*) \leq v \leq f(x^*, y) \ \ \forall\, (x, y) \in C \times D.$$

This says that $v^+ = v^- = v$ and (x^*, y^*) is a saddle point. □

The Von Neumann Minimax Theorem (cont'd)

- Von Neumann's theorem tells us what we need in order to guarantee that our game has a value.

- It is critical that we are dealing with a concave-convex function, and that the strategy sets be convex.

Mixed Strategies

Mixed Strategies

- Von Neumann's theorem suggests that: we need convexity of the sets of strategies, whatever they may be, and convexity-concavity of the payoff function, whatever it may be.
  - A saddle point in pure strategies will not always exist.

- In most two-person zero sum games a saddle point in pure strategies will not exist because that would say that the players should **always** do the same thing.

- A player who chooses a pure strategy randomly chooses a row or column according to some probability process that specifies the chance that each pure strategy will be played. These probability vectors are called **mixed strategies.**

Mixed Strategies (cont'd)

- **Definition 1.3.1** *A mixed strategy is a vector* $X = (x_1, \ldots, x_n)$ *for player I and* $Y = (y_1, \ldots, y_m)$ *for player II, where*

$$x_i \geq 0, \sum_{i=1}^{n} x_i = 1 \quad \textit{and} \quad y_j \geq 0, \sum_{j=1}^{m} y_j = 1.$$

  $x_i =$ *Prob(I uses row i)*, $y_j =$ *Prob(II uses column j)*.

- *Denote the set of mixed strategies with k components by*

$$S_k \equiv \{(z_1, z_2, \ldots, z_k) \mid z_i \geq 0, i = 1, 2, \ldots, k, \sum_{i=1}^{k} z_i = 1\}.$$

  - *A mixed strategy for player I is any element* $X \in S_n$ *and for player II any element* $Y \in S_m$.
  - *A pure strategy* $X \in S_n$ *is an element of the form* X = (0,0,...,0,1,0,...,0), *which represents always playing the row corresponding to the position of the* 1 *in* X.

Expected Payoff

- If the players use mixed strategies, the payoff can be calculated only in the **expected** sense.

- **Definition 1.3.2** *Given a choice of mixed strategy $X \in S_n$ for player I and $Y \in S_m$ for player II, chosen independently, the* **expected** *payoff to player I of the game is*

$$\begin{aligned} E(X,Y) &= \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} Prob(I \text{ uses } i \text{ and } II \text{ uses } j) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} Prob(I \text{ uses } i) P(II \text{ uses } j) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} x_i a_{ij} y_j = X\, A\, Y^T. \end{aligned}$$

Expected Payoff (cont'd)

- *In a zero sum two-person game the expected payoff to player II would be -E(X, Y).*
- *The independent choice of strategy by each player justifies the fact that*

$$Prob(I\ uses\ i\ and\ II\ uses\ j) = Prob(I\ uses\ i)P(II\ uses\ j).$$

- If the game is played only once, player I receives exactly a_{ij}, for the pure strategies *i* and *j* for that play. Only when the game is played many times can player I **expect** to receive approximately *E(X, Y).*

$$E(X, Y) = X\ A\ Y^T = (x_1\ \cdots\ x_n)A_{n\times m}\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

Expected Payoff (cont'd)

- In the mixed matrix zero sum game, the goals now are that player I wants to maximize his expected payoff and player II wants to minimize the expected payoff to I.

- Define the upper and lower values of the mixed game as

$$v^+ = \min_{Y \in S_m} \max_{X \in S_n} XAY^T, \quad \text{and} \quad v^- = \max_{X \in S_n} \min_{Y \in S_m} XAY^T.$$

  - **It is always true that** $v^+ = v^-$ **for the mixed game.**

A Saddle Point in Mixed Strategies

- **Definition 1.3.3** *A* **saddle point in mixed strategies** *is a pair* (X^*, Y^*) *of probability vectors* $X^* \in S_n,\ Y^* \in S_m,$ *which satisfies*

$$E(X, Y^*) \le E(X^*, Y^*) \le E(X^*, Y),\ \ \forall\,(X \in S_n, Y \in S_m).$$

  - *If player I decides to use a strategy other than X* but player II still uses Y*, then I receives an expected payoff smaller than that obtainable by sticking with X *. A similar statement holds for player II.*
  - *So (X*, Y*) is an* **equilibrium** *in this sense.*

A Saddle Point in Mixed Strategies (cont'd)

- A game with matrix *A* have a saddle point in mixed strategies (by Theorem 1.2.3).
  - Define the function $f(X, Y) = E(X, Y) = XAY^T$ and the sets S_n for *X*, and S_m for *Y*.
  - Requirement 1: For any $n \times m$ matrix *A*, this function is concave in *X* and convex in *Y*.
    - It is even **linear** in each variable when the other variable is fixed.
    - **Any linear function is both concave and convex, so our function *f* is concave-convex and certainly continuous.**
  - Requirement 2: The sets S_n and S_m are convex, closed and bounded sets.

The Value of the Game

- **Theorem 1.3.4** *For any n x m matrix A, we have*

$$\min_{Y \in S_m} \max_{X \in S_n} XAY^T = \max_{X \in S_n} \min_{Y \in S_m} XAY^T.$$

  - *The common value is denoted v(A), or value(A), and that is the* **value of the game.**
  - *There is at least one saddle point* $X^* \in S_n,\ Y^* \in S_m$ *so that*

$$E(X, Y^*) \le E(X^*, Y^*) = v(A) \le E(X^*, Y), \textit{ for all } X \in S_n, Y \in S_m.$$

  - **Note that the theorem says <u>there is always at least one saddle point in mixed strategies</u>.**
  - If the game happens to have a saddle point in pure strategies, we should be able to discover that by calculating v^+ and v^- using the columns and rows as did earlier.

Expected Payoff on Strategies

- **Notation** 1.3.5 *For an n x m matrix* $A = (a_{ij})$ *we denote the jth column vector of* A *by* A_j *and the ith row vector of* A *by* ${}_iA$. *So*

$$A_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix} \quad and \quad {}_iA = (a_{i1}, a_{i2}, \cdots, a_{im})$$

  - *If player I decides to use the pure strategy X =* (0, . . . , 0, 1, 0, . . . , 0) *with row i used 100% of the time and player II uses the mixed strategy Y, we denote the expected payoff by* $E(i,Y) = {}_iA \cdot Y^T$.
  - *Similarly, for player II, denote the expected payoff by* $E(X, j) = XA_j$.

$$E(i,Y) = {}_iA \cdot Y^T = \sum_{j=1}^{m} a_{ij} y_j, \;\; E(X,j) = \sum_{i=1}^{n} x_i a_{ij}, \;\; and \;\; E(i,j) = a_{ij}.$$

Expected Payoff on Strategies (cont'd)

- **Lemma 1.3.6** *If* $X \in S_n$ *is any mixed strategy for player I and a is any number so that* $E(X, j) \geq a, \forall j,$ *then for any* $Y \in S_m$, *it is also true that* $E(X,Y) \geq a.$
  - The lemma says that **mixed against all pure is as good as mixed against mixed.**
    - If an inequality holds for a mixed strategy *X* for player I, no matter what column is used for player II, then the inequality holds even if player II uses a mixed strategy.
    - **If *X* is a good strategy for player I when player II uses any pure strategy, then it is still a good strategy for player I even if player II uses a mixed strategy.**
  - Proof:

$$E(X, j) = \sum_i x_i a_{ij} \geq a$$

$$E(X,Y) = \sum_j \sum_i x_i a_{ij} y_j \geq \sum_j a y_j = a, \quad where \quad \sum_j y_j = 1$$

The Value and the Optimal Strategies

Theorem 1.3.7 *Let $A = (a_{ij})$ be an $n \times m$ game with value $v(A)$. Let w be a real number. Let $X^* \in S_n$ be a strategy for player I and $Y^* \in S_m$ be a strategy for player II.*

(a) *If $w \leq E(X^*, j) = X^* A_j = \sum_{i=1}^{n} x_i^* a_{ij}$, $j = 1, \ldots, m$, then $w \leq v(A)$.*

(b) *If $w \geq E(i, Y^*) = {}_iA\, Y^{*T} = \sum_{j=1}^{m} a_{ij} y_j^*$, $i = 1, 2, \ldots, n$, then $w \geq v(A)$.*

(c) *If $E(i, Y^*) = {}_iA\; Y^{*T} \leq w \leq E(X^*, j) = X^* A_j, i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, then $w = v(A)$ and (X^*, Y^*) is a saddle point for the game.*

The Value and the Optimal Strategies (cont'd)

(d) *If* $v(A) \leq E(X^*, j)$ *for all columns* $j = 1, 2, \ldots, m$, *then* X^* *is optimal for player I. If* $v(A) \geq E(i, Y^*)$ *for all rows* $i = 1, 2, \ldots, n$, *then* Y^* *is optimal for player II.*

(e) *A strategy* X^* *for player I is optimal (i.e., part of a saddle point) if and only if* $v(A) = \min_{1 \leq j \leq m} E(X^*, j)$. *A strategy* Y^* *for player II is optimal if and only if* $v(A) = \max_{1 \leq i \leq n} E(i, Y^*)$.

The Value and the Optimal Strategies (cont'd)

- An important way to use the theorem is as a verification tool.
  - If someone says that v is the value of a game and Y is optimal for player II, then you can check it by ensuring that $E(i,Y) \le v$ for every row.
    - If even one of those is not true, then either Y is not optimal for II, or v is not the value of the game.
  - Ex:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad X^* = Y^* = (\frac{1}{2}, \frac{1}{2}), \quad v(A) = \frac{1}{2}.$$

All we have to do is check that

$$E(1, Y^*) = {}_1AY^{*T} = \frac{1}{2} \text{ and } E(2, Y^*) = \frac{1}{2},$$

$$E(X^*, 1) = \frac{1}{2} \text{ and } E(X^*, 2) = \frac{1}{2}.$$

Then the theorem guarantees that $v(A) = \frac{1}{2}$ and (X^*, Y^*) is a saddle point.

The Value and the Optimal Strategies (cont'd)

- If we take $X=(3/4, 1/4)$, then $E(X,2)=1/4 < 1/2$, and so, since $v=1/2$ is the value of the game, we know that X is not optimal for player I.

- Part (c) of the theorem is particularly useful because it gives us a system of inequalities involving $v(A)$, X^*, and Y^*, which, if we can solve them, will give us the value of the game and the saddle points.

Proof of Theorem 1.3.7 (a) (b)

- **(a)** Suppose

$$w \le E(X^*, j) = X^* A_j = \sum_{i=1}^{n} x_i^* a_{ij}, j = 1, 2, \ldots, m.$$

Let $Y^0 = (y_j) \in S_m$ be an optimal mixed strategy for player II. Multiply both sides by y_j and sum on j to see that

$$w = \sum_j y_j w \le \sum_{j=1}^{m} \sum_{i=1}^{n} x_i^* a_{ij} y_j = X^* A Y^{0T} = E(X^*, Y^0) \le v(A),$$

since $\sum_j y_j = 1$, and since $E(X, Y^0) \le v(A)$ for all $X \in S_n$. □

- **(b)** follows in the same way as (a).

Proof of Theorem 1.3.7 (c)

- **(c)** If $\sum_j a_{ij} y_j^* \le w \le \sum_i a_{ij} x_i^*$, we have

$$E(X^*, Y^*) = \sum_i \sum_j x_i^* a_{ij} y_j^* \le \sum_i x_i^* w = w \le \sum_i a_{ij} x_i^*,$$

and

$$E(X^*, Y^*) = \sum_i \sum_j x_i^* a_{ij} y_j^* \ge \sum_j y_j^* w = w \ge E(X^*, Y^*).$$

This says that $w = E(X^*, Y^*)$. So now we have $E(i, Y^*) \le E(X^*, Y^*) \le E(X^*, j)$ for any row i and column j. Taking now any strategies $X \in S_n$ and $Y \in S_m$ and using Lemma 1.3.6, we get $E(X, Y^*) \le E(X^*, Y^*) \le E(X^*, Y)$ so that (X^*, Y^*) is a saddle point and $v(A) = E(X^*, Y^*) = w$. □

Proof of Theorem 1.3.7 (d)

- **(d)** Let $Y^0 \in S_m$ be optimal for player II.
  Then $E(i, Y^0) \le v(A) \le E(X^*, j),$ for all rows *i* and columns *j*, where the first inequality comes from **the definition of optimal for player II.**
- Now use part (c) of the theorem to see that *X** is optimal for player I. The second part of (d) is similar. □

Proof of Theorem 1.3.7 (e)

- **(e)** Begin by establishing that $\min_Y E(X,Y) = \min_j E(X, j)$ for any fixed $X \in S_n$. To see this, since every pure strategy is also a mixed strategy, it is clear that $\min_Y E(X,Y) \le \min_j E(X, j)$. Now set $a = \min_j E(X, j)$. Then

$$0 \le \min_{Y \in S_m} \sum_j (E(X.j) - a) y_j = \min_{Y \in S_m} E(X,Y) - a,$$

since $E(X, j) \ge a$ for each *j* = 1,2,...,m. Consequently, $\min_Y E(X,Y) \ge a$, and putting the two inequalities together, we conclude that $\min_Y E(X,Y) = \min_j E(X, j)$.

Proof of Theorem 1.3.7 (e) (cont'd)

Using the definition *of v(A),* we then have

$$v(A) = \max_X \min_Y E(X,Y) = \max_X \min_j E(X,j).$$

We can also show that $v(A) = \min_Y \max_i E(i,Y)$. Consequently,

$$v(A) = \max_{X \in S_n} \min_{1 \le j \le m} E(X,j) = \min_{Y \in S_m} \max_{1 \le i \le n} E(i,Y).$$

If *X** is optimal for player I, then

$$v(A) = \max_X \min_Y E(X,Y) \le \min_Y E(X^*,Y) = \min_j E(X^*,j).$$

If $v(A) \le \min_j E(X^*, j)$, then $v(A) \le E(X^*, j)$ for any column, and so $v(A) \le E(X^*, Y)$ for any $Y \in S_m$, by Lemma 1.3.6, which implies that *X** is optimal for player I. □

The Value and the Optimal Strategies (cont'd)

- **Corollary 1.3.8** $v(A) = \min_{Y \in S_m} \max_{1 \le i \le n} E(i, Y) = \max_{X \in S_n} \min_{1 \le j \le m} E(X, j).$ In addition, $v^- = \max_i \min_j a_{ij} \le v(A) \le \min_j \max_i a_{ij} = v^+.$
  - Be aware of the fact that not only are the min and max in the corollary being switched but also the sets over which the min and max are taken are changing.

The Value and the Optimal Strategies (cont'd)

- Consider the system of inequations

$$E(X, j) \geq v, j = 1, \ldots, m, \text{ for the unknowns } X = (x_1, \ldots, x_n),$$

along with the condition $x_1 + \ldots + x_n = 1.$

  - We need the last equation because *v* is also an unknown.
  - If we can solve these inequalities and the x_i variables turn out to be nonnegative, then that gives us a candidate for the optimal mixed strategy for player I, and our candidate for the value *v = v(A).*
  - Once we know, or think we know *v(A),* then we can solve the system $E(i, Y) \leq v(A)$ for player II's *Y* strategy.
  - If all the variables y_j are nonnegative and sum to one, then part (c) of Theorem 1.3.7 tells us that we have the solution and we are done.

Example 1.9

- Game matrix A, with $v^- = -1 \quad and \quad v^+ = 3$, no saddle point in pure strategies.

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 9 \end{bmatrix}$$

- Use parts (c) and (e) of Theorem 1.3.7 to find the mixed saddle.
  - Suppose that *X* = (*x,1-x*) is optimal and *v* = *v(A)* is the value of the game.
  - Then *v≤E*(*X,* 1) and *v≤E*(*X,* 2), which gives us *v≤4x-1* and *v≤*-10x+9.
  - Solve $v = 4x - 1 \quad and \quad v = -10x + 9, \quad get \quad x = {}^{10}\!/_{14} \quad and \quad v = {}^{26}\!/_{14}.$
  - $X = ({}^{10}\!/_{14}, {}^{4}\!/_{14})$ is a legitimate strategy and *v* satisfies the conditions in Theorem 1.3.7, we know that *X* is optimal. Similarly $Y = ({}^{10}\!/_{14}, {}^{4}\!/_{14}).$

Example 1.10

- Game matrix

Evens	Odds		
I/II	1	2	3
1	1	−1	1
2	−1	1	−1
3	1	−1	1

  – $v^- = -1 \;\; and \;\; v^+ = +1$, this game does not have a saddle point using only pure strategies.

- Find the mixed saddle point
  – Suppose that *v* is the value of this game and $X^* = (x_1, x_2, x_3)$, $Y^* = (y_1, y_2, y_3)$ is a saddle point.

Example 1.10 (cont'd)

– According to Theorem 1.3.7, these quantities should satisfy

$$E(i, Y^*) = {}_iA\,Y^{*T} = \sum_{j=1}^{3} a_{ij} y_j \leq v \leq E(X^*, j) = X^* A_j = \sum_{i=1}^{3} x_i a_{ij}.$$

– Using the values from the matrix, we have the system of inequalities

$$y_1 - y_2 + y_3 \leq v, \ -y_1 + y_2 - y_3 \leq v, \quad \text{and} \ y_1 - y_2 + y_3 \leq v,$$

$$x_1 - x_2 + x_3 \geq v, \ -x_1 + x_2 - x_3 \geq v, \quad \text{and} \ x_1 - x_2 + x_3 \geq v.$$

– With $x_1 + x_2 + x_3 = 1$,

$$1 - 2x_2 \geq v \quad \text{and} \quad -1 + 2x_2 \geq v \implies -v \geq 1 - 2x_2 \geq v.$$

– Assume $v = 0$ and $x_2 = \frac{1}{2}$, then $x_1 + x_3 = \frac{1}{2}$.

– Since row 3 (or row 1) is a redundant strategy, If we drop row 3 we perform the same set of calculations but we quickly find that $x_2 = \frac{1}{2} = x_1$.

Example 1.10 (cont'd)

- We assumed that $v \geq 0$ to get this but now we have our candidates for the saddle points and value, namely, $v = 0,\ X^* = (\frac{1}{2}, \frac{1}{2}, 0)$ and also, in a similar way $Y^* = (\frac{1}{2}, \frac{1}{2}, 0)$.
- There are an infinite number of saddle points, $X^* = (x_1, \frac{1}{2}, \frac{1}{2} - x_1)$, $0 \leq x_1 \leq \frac{1}{2}$, and $Y^* = (y_1, \frac{1}{2}, \frac{1}{2} - y_1), 0 \leq y_1 \leq \frac{1}{2}$.
  Nevertheless, there is only one value for this, or any matrix game, and it is $v = 0$ in the game of odds and evens.

Properties of Optimal Strategies

1. If w is any number such that $E(i,Y) \leq w \leq E(X,j), i = 1,\ldots,n, j = 1,\ldots,m$, where X is a strategy for player I and Y is a strategy for player II, then $w = value(A)$ and (X,Y) must be a saddle point. This is the way to check whether you have a solution to the game. This is part (c) of Theorem 1.3.7 but worth repeating.

2. If X is a strategy for player I and $value(A) \leq E(X,j), j = 1,\ldots,n$, then X is optimal for player I. If Y is a strategy for player II and $value(A) \geq E(i,Y), i = 1,\ldots,m$, then Y is optimal for player II.

Properties of Optimal Strategies (cont'd)

3. If Y is optimal for II and $y_j > 0$, then $E(X, j) = value(A)$ for any optimal mixed strategy X for I. Similarly, if X is optimal for I and $x_i > 0$, then $E(i, Y) = value(A)$ for any optimal Y for II. Thus, if any optimal mixed strategy for a player has a strictly positive probability of using a row or a column, then that row or column played against any optimal opponent strategy will yield the value. This result is also called the **Equilibrium Theorem**.

4. If X is any optimal strategy for player I and $E(X, j) > value(A)$ for some column j, then for any optimal strategy Y for player II, we must have $y_j = 0$. Player II would never use column j in any optimal strategy for player II. Similarly, if Y is any optimal strategy for player II and $E(i, Y) < value(A)$, then any optimal strategy X for player I must have $x_i = 0$. If row i for player I gives a payoff when played against an optimal strategy for player II strictly below the value of the game, then player I would never use that row in any optimal strategy for player I.

Properties of Optimal Strategies (cont'd)

5. If for any optimal strategy Y for player II, $y_j = 0$, then there is an optimal strategy X for player I so that $E(X, j) > value(A)$. If for any optimal strategy X for I, $x_i = 0$, then there is an optimal strategy Y for II so that $E(i, Y) < value(A)$. This is the converse statement to property 4.

6. If player I has more than one optimal strategy, then player I's set of optimal strategies is a convex, closed, and bounded set. Also, if player II has more than one optimal strategy, then player II's set of optimal strategies is a convex, closed, and bounded) set.

Properties of Optimal Strategies (cont'd)

- Remarks
  - These properties and Theorem 1.3.7 give us a way of solving games algebraically without having to solve inequalities.
  - The value of the game and the optimal strategies *X** and *Y** must satisfy $E(i, Y^*) = v(A)$ for each row with $x_i^* > 0$ and $E(X^*, j) = v(A)$ for every column j with $y_j^* > 0$.

Properties of Optimal Strategies (cont'd)

Proof of Property 4. If it happens that (X^*, Y^*) are optimal and there is a component of $X^* = (x_1, \ldots, x_k^*, \ldots, x_n)$, say, $x_k^* > 0$ but $E(k, Y^*) < v(A)$, then multiplying both sides of $E(k, Y^*) < v(A)$ by x_k^* yields $x_k^* E(k, Y^*) < x_k^* v(A)$. Now, it is always true that for any row $i = 1, 2, \ldots, n$,

$$E(i, Y^*) \leq v(A), \text{ which implies that } x_i E(i, Y^*) \leq x_i v(A).$$

But then, because $v(A) > E(k, Y^*)$ and $x_k^* > 0$, by adding, we get

$$\sum_{i=1, i \neq k}^{n} x_i E(i, Y^*) + x_k^* E(k, Y^*) = \sum_{i=1}^{n} x_i E(i, Y^*) < \sum_{i=1}^{n} x_i v(A) = v(A).$$

Properties of Optimal Strategies (cont'd)

We see that, under the assumption $E(k, Y^*) < v(A)$, we have

$$v(A) = E(X^*, Y^*) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i a_{ij} y_j = \sum_{i=1}^{n} x_i E(i, Y^*) < v(A),$$

which is a contradiction. But this means that if $x_k^* > 0$ we must have $E(k, Y^*) = v(A)$. $\square$

Example 1.11

- Consider the game matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

  – Conjecture that if $X = (x_1, x_2, x_3)$ is optimal, then $x_i > 0$. By property 3 for $Y = (y_1, y_2, y_3)$

$$E(1, Y) = 1y_1 + 2y_2 + 3y_3 = v$$
$$E(2, Y) = 3y_1 + 1y_2 + 2y_3 = v$$
$$E(3, Y) = 2y_1 + 3y_2 + 1y_3 = v$$
$$y_1 + y_2 + y_3 = 1.$$

Example 1.11 (cont'd)

– Obtain the solution $y_1 = y_2 = y_3 = \frac{1}{3}$, and $v = 2$.
– Theorem 1.3.7 guarantees that $Y = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is indeed an optimal mixed strategy for player II and *v*(*A*) = 2 is the value of the game. A similar approach proves that $X = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is also optimal for player I.
– Maple commands for getting the solution:

```
> eqs:={y1+2*y2+3*y3-v=0,
        3*y1+y2+2*y3-v=0,
        2*y1+3*y2+y3-v=0,
        y1+y2+y3-1=0};
> solve(eqs,[y1,y2,y3,v]);
```

Example 1.12

- Consider the game matrix

$$A = \begin{bmatrix} -2 & 2 & -1 \\ 1 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix}$$

  - A saddle point at $X^* = (0, 1, 0), Y^* = (0, 0, 1)$, and $v(A) = 1$.
  - If we assumed that *X* is optimal and $x_i > 0, i = 1, 2, 3,$ then it would have to be true that

$$\begin{aligned} -2y_1 + 2y_2 - y_3 &= 1 \\ y_1 + y_2 + y_3 &= 1 \\ 3y_1 + y_3 &= 1 \end{aligned}$$

    because we know that *v* = 1 . But there is only one solution of this system, $Y = (\frac{2}{5}, \frac{4}{5}, -\frac{1}{5}),$ which is not a strategy. This means that our assumption about the existence of an optimal strategy *X* for player I must be wrong.

Example 1.12 (cont'd)

- For $X^* = (0, 1, 0)$, $E(2, Y) = 1$, $E(1, Y) < 1$, and $E(3, Y) < 1$. We need to look for y_1, y_2, y_3 so that

$$y_1 + y_2 + y_3 = 1, \; -2y_1 + 2y_2 - y_3 < 1, 3y_1 + y_3 < 1.$$

- Replace $y_3 = 1 - y_1 - y_2$ and then get a graph of the region of points satisfying all the inequalities in (y_1, y_2) space in Figure 1.5.
- There are lots of points which work. In particular, *Y* = (0.15,0.5,0.35) will give an optimal strategy for player II in which all $y_j > 0$.

Example 1.12 (cont'd)

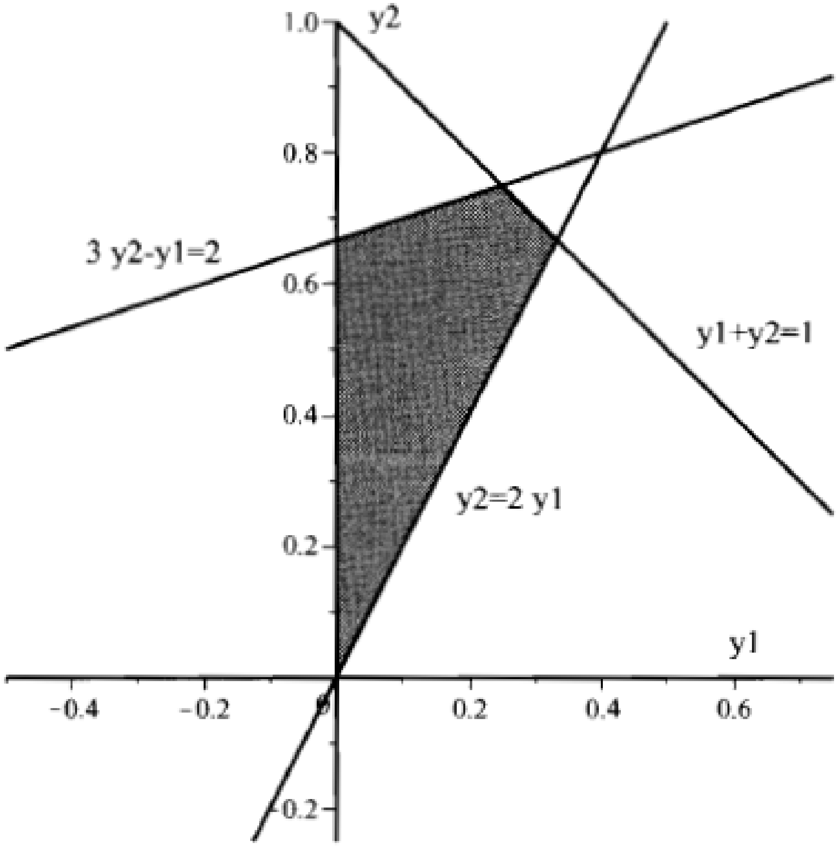


Figure 1.5 Optimal strategy set for *Y.*

Dominated Strategies

- Sometimes we can reduce the size of the matrix *A* by eliminating rows or columns (i.e., strategies) that will never be used because there is always a better row or column to use. This is elimination by **dominance.**
  - If we can reduce it to a 2 x *m* or *n* x 2 game, we can solve it by a graphical procedure. If we can reduce it to a 2 x 2 matrix, we can use the formulas (following on).

Dominated Strategies (cont'd)

- **Definition 1.3.9** *Row i dominates row k if* $a_{ij} \geq a_{kj}$ *for all* $j = 1, 2, \ldots, m.$ *This allows us to remove row k. Column j dominates column k if* $a_{ij} \leq a_{ik}, i = 1, 2, \ldots, n.$ *This allows us to remove column k. Strict dominance means the inequalities are strict in at least one payoff pair in a row or a column.*

- **Remark.** A row that is dropped because it is **strictly dominated** is played in a mixed strategy with probability 0. But a row that is dropped because it is equal to another row may not have probability 0 of being played.

Dominated Strategies (cont'd)

- For example, suppose that we have a matrix with three rows and row 2 is the same as row 3. If we drop row 3, we now have two rows and the resulting optimal strategy will look like $X^* = (x_1, x_2)$.
- For the original game the optimal strategy could be $X^* = (x_1, x_2, 0)$ or $X^* = (x_1, x_2/2, x_2/2)$, or in fact $X^* = (x_1, \lambda x_2, (1-\lambda)x_2)$ for any $0 \leq \lambda \leq 1$, and this is the most general description.
- A duplicate row is a redundant row and may be dropped to reduce the size of the matrix. **But you must account for redundant strategies.**

Dominated Strategies (cont'd)

- Another way to reduce the size of a matrix is to drop rows or columns by dominance through a <u>convex combination</u> of other rows or columns.
  - If there is a constant $\lambda \in [0, 1]$ so that

$$a_{kj} \le \lambda a_{pj} + (1 - \lambda) a_{qj}, \quad j = 1, \ldots, m,$$

    then row *k* is dominated and can be dropped.
  - Similarly, column *k* is dominated by a convex combination of columns *p* and *q* if

$$a_{ik} \ge \lambda a_{ip} + (1 - \lambda) a_{iq}, \quad i = 1, \ldots, n.$$

Example 1.13

- Consider the 3x4 game

$$A = \begin{bmatrix} 10 & 0 & 7 & 4 \\ 2 & 6 & 4 & 7 \\ 5 & 2 & 3 & 8 \end{bmatrix}.$$

  - We may drop column 4 right away because every number in that column is larger than each corresponding number in column 2. So now we have

$$\begin{bmatrix} 10 & 0 & 7 \\ 2 & 6 & 4 \\ 5 & 2 & 3 \end{bmatrix}.$$

Example 1.13 (cont'd)

- Row 3 is dominated by a convex combination of rows 1 and 2. If that is true, we must have, for some $0 \le \lambda \le 1$, the inequalities

  $$5 \le \lambda(10) + (1-\lambda)(2),\ 2 \le 0(\lambda) + 6(1-\lambda),\ 3 \le 7(\lambda) + 4(1-\lambda).$$

  Simplifying, $5 \le 8\lambda + 2, 2 \le 6 - 6\lambda, 3 \le 3\lambda + 4$. This says any $\frac{3}{8} \le \lambda \le \frac{2}{3}$ will work. So now the new matrix is

  $$\begin{bmatrix} 10 & 0 & 7 \\ 2 & 6 & 4 \end{bmatrix}.$$

- Column 3 might be dominated by a combination of columns 1 and 2.

  $$7 \ge 10\lambda + 0(1-\lambda) = 10\lambda, \text{ and } 4 \ge 2\lambda + 6(1-\lambda) = -4\lambda + 6.$$

  Require that $\frac{1}{2} \le \lambda \le \frac{7}{10}$.

Example 1.13 (cont'd)

– Finally, we are down to a 2 x 2 matrix

$$\begin{bmatrix} 10 & 0 \\ 2 & 6 \end{bmatrix}.$$

– Solve these small games graphically, or by assuming that each row and column will be used with positive probability and then solving the system of equations.

Solution:

$$v(A) = \frac{30}{7}$$
$$X^* = (\tfrac{2}{7}, \tfrac{5}{7}, 0)$$
$$Y^* = (\tfrac{3}{7}, \tfrac{4}{7}, 0, 0).$$

Solving 2 × 2 Games Graphically

Solving 2×2 Games Graphically

- Consider the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}.$$

  - We must check firstly whether there are pure optimal strategies because if there are, then we can't use the graphical method. Since $v^- = 2$ and $v^+ = 3$, we know the optimal strategies must be mixed.
  - Use Theorem 1.3.7 part (c) to find the optimal strategy and the value.

$$E(X,1) = XA_1 = x + 3(1-x) \text{ and } E(X,2) = XA_2 = 4x + 2(1-x).$$

We plot each of these functions of *x* on the same graph in Figure 1.6. Each plot will be a straight line with $0 \le x \le 1$.

Solving 2 × 2 Games Graphically (cont'd)

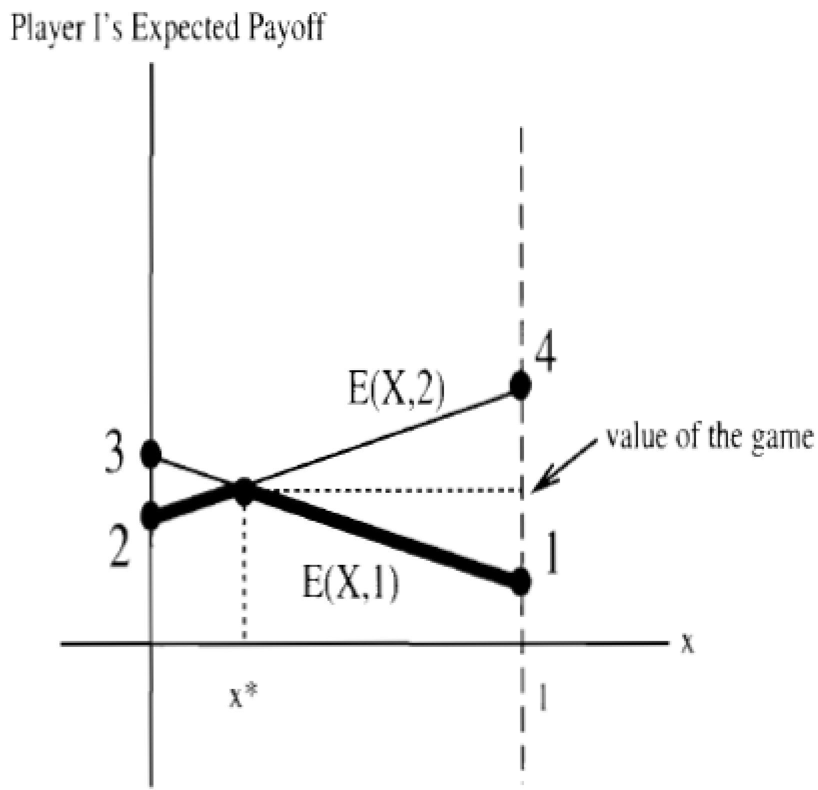


Figure 1.6 *X* against each column for player II.

Analysis of the Graph

- Analysis
  - The point at which the two lines intersect is $(x^* = \frac{1}{4}, \frac{10}{4})$.
  - If player I chooses an $x < x^*$, then the best I can receive is on the highest line when x is on the left of x^*.
    - $E(X,1) = x + 3(1-x) > \frac{10}{4}$.
    - Player I will receive this higher payoff only if player II decides to play column 1.
    - If player II use column 2, then I would receive a payoff on the **lower** line $E(X,2) < \frac{10}{4}$.
  - If player I chooses an $x > \frac{1}{4}$
    - $E(X,2) > \frac{10}{4}$.
    - The best I could get would happen if player II chose to use column 2.
    - If player II use column 1, then I would receives some payoff on the line $E(X,1) < \frac{10}{4}$.

Analysis of the Graph (cont'd)

- Conclusion
  - Player I, assuming that player II will be doing her best, will choose to play $X = (x^*, 1 - x^*) = (\frac{1}{4}, \frac{3}{4})$ and then receive exactly the payoff $v(A) = \frac{10}{4}$.
  - Player I will rationally choose the maximum minimum. The minimums are the bold lines and the maximum minimum is at the intersection, which is the highest point of the bold lines.
  - Player I will choose a mixed strategy so that she will get $\frac{10}{4}$ no matter what player II does, and if II does not play optimally, player I can get more than $\frac{10}{4}$.

Graphical Solution of $2 \times m$ and $n \times 2$ Games

Graphical Solution of 2 × *m* Games

- **Assuming that there is no pure saddle,** i.e., $v^+ > v^-$, consider the matrix A, and denote A_j the *j*th column, ${}_iA$ the *i*th row.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \end{bmatrix}$$

  – Suppose that player I chooses a mixed strategy $X = (x, 1-x)$, $0 < x < 1$, and player II chooses column *j.*
  The payoff to player I is $E(X, j) = XA_j$ or, written out

$$E(X, j) = x\, a_{1j} + (1-x)a_{2j}.$$

  – A mixed strategy is determined by the choice of the single variable $x \in [0, 1]$. This is perfect for drawing a plot.

Graphical Solution of $2 \times m$ Games (cont'd)

- On a graph (with *x* on the horizontal axis), $y = E(X, j)$ is a straight line through the two points $(0, a_{2j})$ and $(1, a_{1j})$. For each column *j*,

$$f(x) = \min_{1 \le j \le m} X\, A_j = \min_{1 \le j \le m} x\, a_{1j} + (1-x)a_{2j}.$$

  This is called the **lower envelope** of all the straight lines associated to each strategy *j* for player II. Then let $0 \le x^* \le 1$ be the point where the maximum of *f* is achieved:

$$f(x^*) = \max_{0 \le x \le 1} f(x) = \max_x \min_{1 \le j \le m} x\, a_{1j} + (1-x)a_{2j} = \max_x \min_j E(X, j).$$

- Then $X^* = (x^*, 1 - x^*)$ is the optimal strategy for player I and $f(x^*)$ will be the value of the game *v*(*A*). This is shown in Figure 1.7 for a 2 x 3 game.

Graphical Solution of $2 \times m$ Games (cont'd)

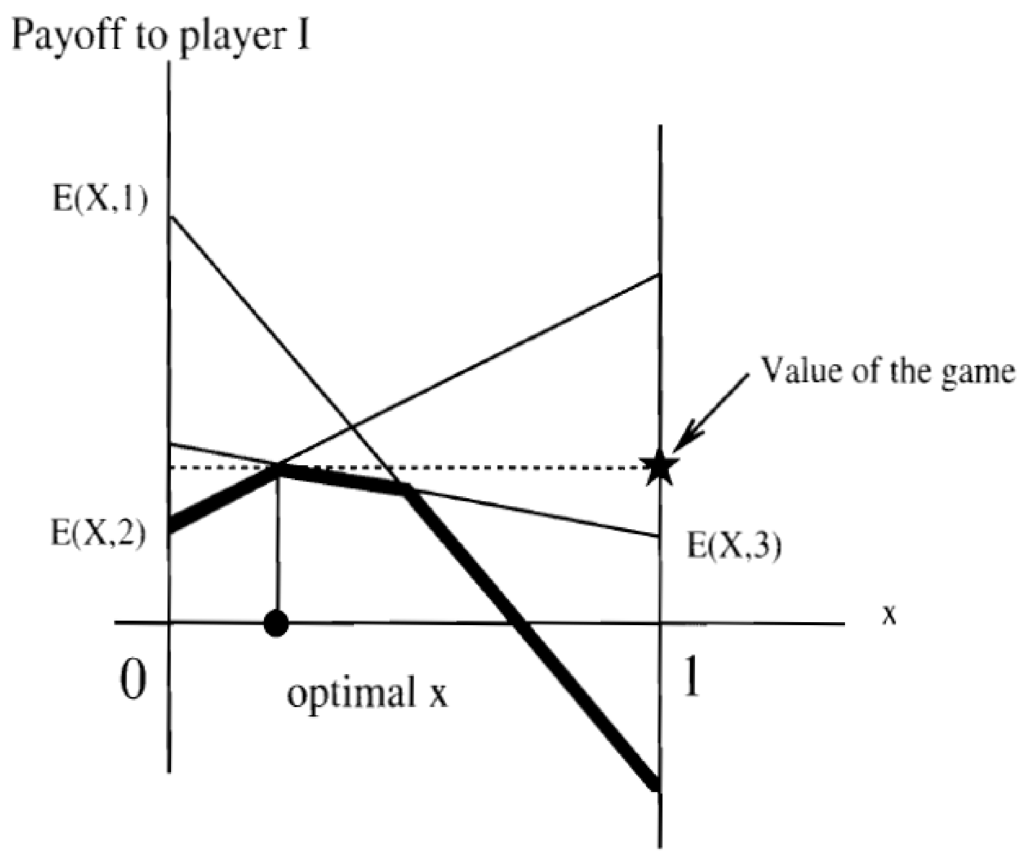


Figure 1.7 Graphical solution for 2 x 3 game.

Graphical Solution of $2 \times m$ Games (cont'd)

– Each line represents the payoff that player I would receive by playing the mixed strategy $X = (x, 1-x)$, with player II always playing a fixed column.

- If player I decides to play the mixed strategy $X_1 = (x_1, 1-x_1)$ where x_1 is to the left of the optimal value, then player II would choose to play column 2.
- If player I decides to play the mixed strategy $X_2 = (x_2, 1-x_2)$, where x_2 is to the right of the optimal value, then player II would choose to play column 3, up to the point of intersection where $E(X, 1) = E(X, 3)$, and then switch to column 1.
- **Player I would choose the *x* that guarantees that she will receive the maximum of all the lower points of the lines.**

Graphical Solution of $2 \times m$ Games (cont'd)

- By choosing this optimal value, say, x*, it will be the case that player II would play some combination of columns 2 and 3.
  - It would be a mixture (a convex combination) of the columns because if player II always chose to play, say, column 2, then player I could do better by changing her mixed strategy to a point to the right of the optimal value.
- For finding the optimal strategy for player II, the only two columns being used in an optimal strategy for player I are columns 2 and 3.
  - By the properties of optimal strategies (1.3.1), that for this particular graph we can eliminate column 1 and reduce to a 2 x 2 matrix.

Example 1.14

- Consider the payoff matrix and the graph for player I:

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \end{bmatrix}$$

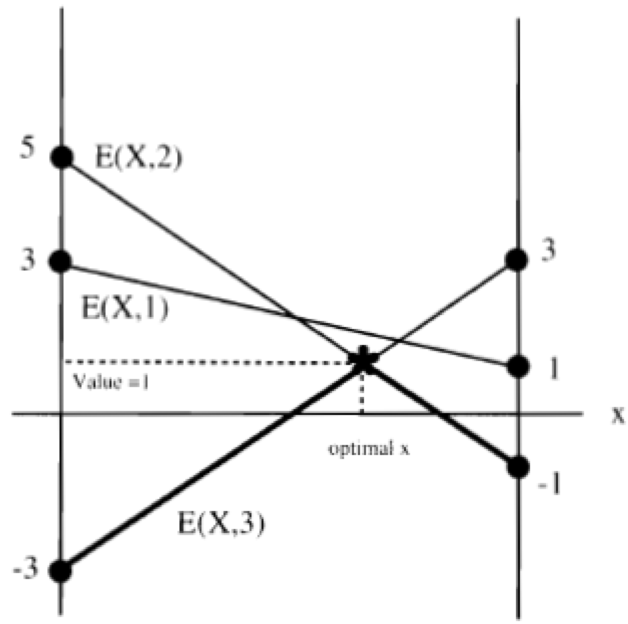


Figure 1.8 Mixed for player I versus player II's columns.

Example 1.14 (cont'd)

- The optimal strategy for I is the *x* value where the two lower lines intersect and yields $X^* = (\frac{2}{3}, \frac{1}{3})$. Also, $v(A) = E(X^*, 3) = E(X^*, 2) = 1.$
- The figure indicates that column 1 is dominated by columns 2 and 3 because it is always above the optimal point. $1 \geq -\lambda + 3(1-\lambda)$ and $3 \geq 5\lambda - 3(1-\lambda)$ imply that for $\frac{1}{2} \leq \lambda \leq \frac{3}{4}$ column 1 may be dropped.
- Now consider the subgame with the first column removed:

$$A1 = \begin{bmatrix} -1 & 3 \\ 5 & -3 \end{bmatrix}.$$

Example 1.14 (cont'd)

- Solve this graphically for player II assuming that II uses $Y = (y, 1-y)$. Consider the payoffs $E(1,Y)$ and $E(2,Y)$.
- Player II wants to choose *y* so that no matter what I does she is guaranteed the smallest maximum. This is now the lowest point of the highest part of the lines in Figure 1.9.
- The lines intersect with $y^* = \frac{1}{2}$.
- The optimal strategy for II is $Y^* = (0, \frac{1}{2}, \frac{1}{2})$, and the value $v(A) = 1$.

Example 1.14 (cont'd)

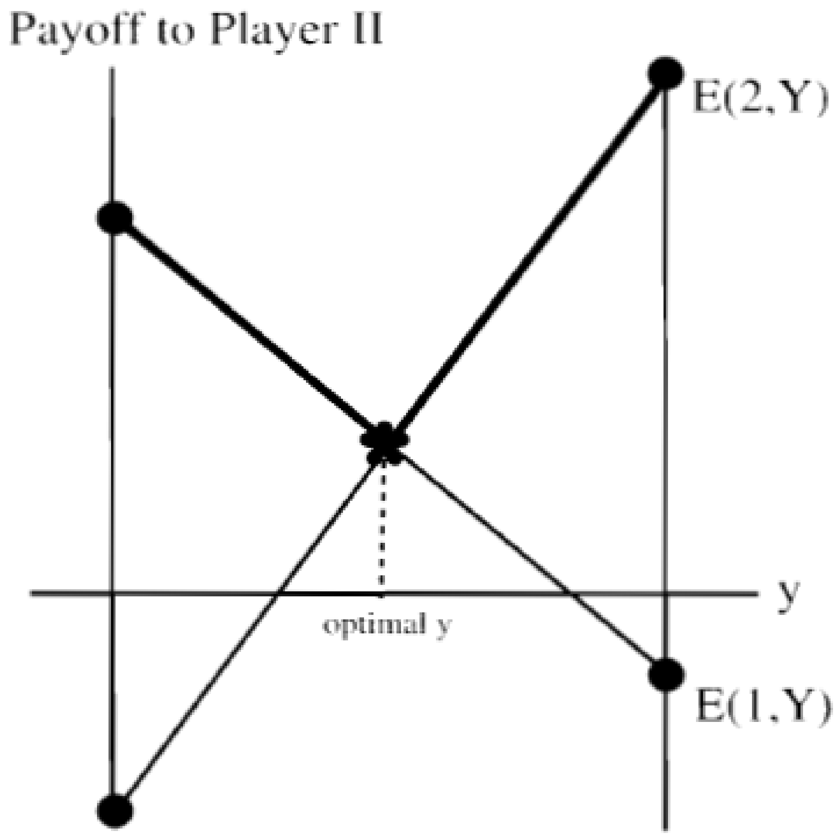


Figure 1.9 Mixed for player II versus I's rows.

Graphical Solution of $n \times 2$ Games

- Consider an $n \times 2$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ \vdots & \vdots \\ a_{n1} & a_{n2} \end{bmatrix}$$

Assume that player II uses the mixed strategy $Y = (y, 1-y)$, $0 < y < 1$. Then II wants to choose *y* to **minimize** the quantity

$$\max_{1 \le i \le n} E(i, Y) = \max_{1 \le i \le n} {}_iAY^T = \max_{1 \le i \le n} y(a_{i1}) + (1-y)(a_{i2}).$$

Graphical Solution of $n \times 2$ Games (cont'd)

- The graph of the payoffs (to player I) $E(i, Y)$ will be a straight line.
- Player I will want to go as high as possible; Player II will play the mixed strategy Y, which will give the lowest maximum.
- The optimal y^* will be the point giving the minimum of the upper envelope.

Example 1.15

- Consider

$$A = \begin{bmatrix} -1 & 2 \\ 3 & -4 \\ -5 & 6 \\ 7 & -8 \end{bmatrix}$$

This is a 4 x 2 game without a saddle point in pure strategies since $v^- = -1, v^+ = 6.$ Try to solve the game graphically.

  - Suppose that player II uses the strategy $Y = (y, 1-y),$ then we graph the payoffs $E(i, Y), i = 1, 2, 3, 4,$ as shown in Figure 1.10.
  - The optimal strategy for *Y* will be determined at the intersection point of $E(4,Y) = 7y - 8(1-y)$ and $E(1,Y) = -y + 2(1-y)$. This occurs at the point $y^* = \frac{5}{9}$ and the corresponding value of the game will be $v(A) = \frac{1}{3}.$ The optimal strategy for player II is $Y^* = (\frac{5}{9}, \frac{4}{9}).$

Example 1.15 (cont'd)

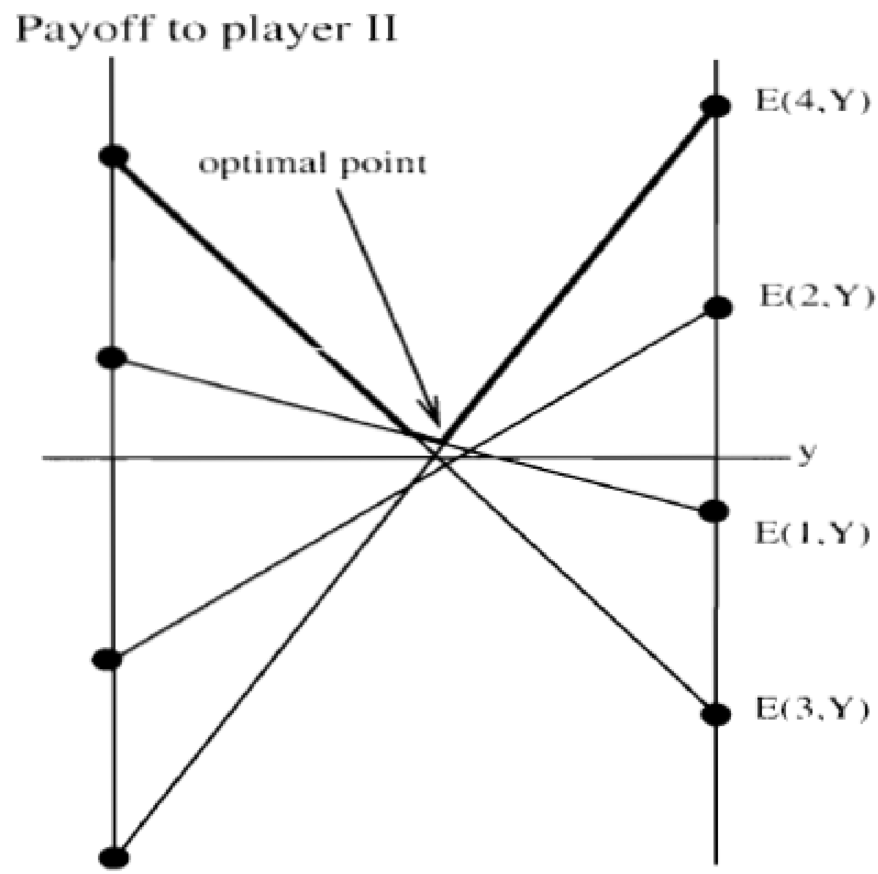


Figure 1.10 Mixed for player II versus 4 rows for player I.

Example 1.15 (cont'd)

- Since $3 \le 7\frac{1}{2} - 1\frac{1}{2}$ and $-4 \le -8\frac{1}{2} + 2\frac{1}{2}$, row 2 is dominated by a convex combination of rows 1 and 4; so row 2 may be dropped.
- Row 3 is dropped because its payoff line $E(3, Y)$ does not pass through the optimal point.
- Considering the matrix using only rows 1 and 4, we now calculate $E(1, X) = -x + 7(1 - x)$ and $E(4, X) = 2x - 8(1 - x)$ which intersect at $(x = \frac{5}{6}, \frac{1}{3})$.
- We obtain that row 1 should be used with probability $\frac{5}{6}$ and row 4 should be used with probability $\frac{1}{6}$, so $X^* = (\frac{5}{6}, 0, 0, \frac{1}{6})$. $v(A) = \frac{1}{3}$.
- In the above, we drop rows 2 and 3 to find the optimal strategy for player 1. In general, we may drop the rows (or columns) not used to get the optimal intersection point. ➔ ***This is not always true !***

Example 1.15 (cont'd)

- A verification that these are indeed optimal uses Theorem 1.3.7(c). We check that $E(i, Y^*) \le v(A) \le E(X^*, j)$. This gives

$$\begin{bmatrix} \frac{5}{6} & 0 & 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & -4 \\ -5 & 6 \\ 7 & -8 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

and $$\begin{bmatrix} -1 & 2 \\ 3 & -4 \\ -5 & 6 \\ 7 & -8 \end{bmatrix} \begin{bmatrix} \frac{5}{9} \\ \frac{4}{9} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{9} \\ -\frac{1}{9} \\ \frac{1}{3} \end{bmatrix}.$$

Example 1.16

- Poker game rules
  - Player I is dealt a card that may be an ace or a king. Player I sees the result but II does not. Player I may then choose to fold or bet.
  - If I folds, he has to pay player II $1. If I bets, player II may choose to fold or call.
  - If II folds, she pays player I $1. If player II calls and the card is a king, then player I pays player II $2, but if the card comes up ace, then player II pays player I $2.
  - I must pay II $1 when I gets a king and he folds.
  - Player I is hoping that player II will fold if I bets while holding a king. This is the element of bluffing, because if II calls while I is holding a king, then I must pay II $2.

Example 1.16 (cont'd)

- Strategies
  - *FF:* Fold on ace and fold on king
  - *FB:* Fold on ace and bet on King
  - *BF:* Bet on ace and fold on king
  - *BB:* Bet on ace and bet on king
  - Player II has only two strategies, namely, *F* (fold) or *C* (call).
  - Assuming that the probability of being dealt a king or an ace is $\frac{1}{2}$.

I/II	C	F
FF	-1	-1
FB	$-\frac{3}{2}$	0
BF	$\frac{1}{2}$	0
BB	0	1

Example 1.16 (cont'd)

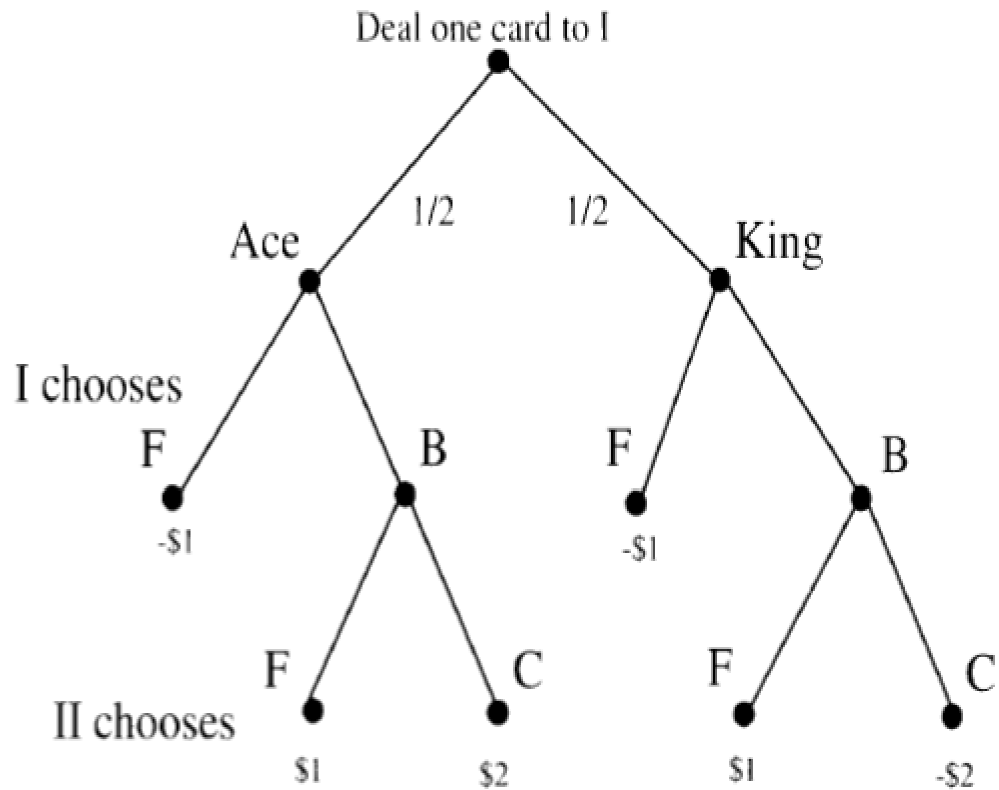


Figure 1.11 A simple poker game. F=fold, B=bet, C=call.

Example 1.16 (cont'd)

- If I plays *BF* and II plays C, this means that player I will bet if he got an ace, and fold if he got a king. Player II will call no matter what. We calculate the expected payoff to I as $\frac{1}{2}\cdot 2+\frac{1}{2}(-1)=\frac{1}{2}$. Similarly,

$$E(FB,F)=\frac{1}{2}(-1)+\frac{1}{2}\cdot 1=0,$$
$$\text{and}\quad E(FB,C)=\frac{1}{2}(-1)+\frac{1}{2}(-2)=-\frac{3}{2},$$

and so on. This is a 4 x 2, game which we can solve graphically.

Solve the 4 x 2 Poker Game Graphically

- 1. Dominance
  - The lower and upper values are $v^- = 0, v^+ = \frac{1}{2},$ so there is no saddle point in pure strategies.
  - Row 1, namely *FF*, is a strictly dominated strategy, so we may drop it. It is never worth it to player I to simply fold.
  - Row 2 is also strictly dominated by row 4 and can be dropped.
  - So we are left with considering the 2 x 2 matrix

$$A' = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}.$$

Solve the 4 x 2 Poker Game Graphically (cont'd)

- 2. Optimal strategy for player II
  - Suppose that II plays $Y = (y, 1-y)$. Then

$$E(BF, Y) = A'Y^T = \frac{1}{2}\, y \text{ and } E(BB, Y) = (1-y).$$

  - Two lines intersect at $\frac{1}{2}y = 1 - y$, so that $y^* = \frac{2}{3}$.
  - The optimal strategy for II is $Y^* = (\frac{2}{3}, \frac{1}{3})$, so II should call two-thirds of the time and bet one-third of the time.
  - The value of the game is then at the point of intersection $v = \frac{1}{3}$.
  - Player II is at a distinct disadvantage since the value of this game is $v = \frac{1}{3}$. Player II in fact would never be induced to play the game unless player I pays II exactly $\frac{1}{3}$ before the game begins. That would make the value zero and hence a fair game.

Solve the 4 x 2 Poker Game Graphically (cont'd)

- 3. Optimal strategy for player I
  - Suppose that I plays $X = (x, 1 - x)$. Then

$$E(X, C) = XA' = \frac{1}{2}x \text{ and } E(X, F) = 1 - x.$$

  - There are only two lines, we again calculate the intersection point and obtain the optimal strategy for I as $X^* = (0, 0, \frac{2}{3}, \frac{1}{3})$.
  - A interesting phenomenon that the optimal strategy for player I has him betting one-third of the time when he has a losing card (king).
  - Bluffing with positive probability is a part of an optimal strategy when done in the right proportion.

Solve the 4 x 2 Poker Game Graphically (cont'd)

- Player II in fact would never be induced to play the game unless player I pays II exactly 1/3 before the game begins.
  - That would make the value zero and hence a fair game.

Best Response Strategies

Definition of Best Response Strategy

- **Definition 1.6.1** *A mixed strategy X* for player I is a* **best response strategy** *to the strategy Y for player II if it satisfies*

$$\max_{X \in S_n} E(X,Y) = \max_{X \in S_n} \sum_{i=1}^{n} \sum_{j=1}^{m} x_i^* a_{ij} y_j = E(X^*, Y).$$

A mixed strategy Y for player II is a best response strategy to the strategy X for player I if it satisfies*

$$\min_{Y \in S_m} E(X,Y) = \min_{Y \in S_m} \sum_{i=1}^{n} \sum_{j=1}^{m} x_i a_{ij} y_j^* = E(X, Y^*).$$

  - If (*X**, *Y**) is a saddle point of the game, then *X** is the best response to Y*, and vice versa.
  - Unfortunately, knowing this doesn't provide a good way to calculate *X** and *Y** because they are **both** unknown at the start.

Example 1.17

- Consider the 3 x 3 game

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

The saddle point is $X^* = (0, \frac{1}{2}, \frac{1}{2}) = Y^*$ and $v(A) = 1.$

Suppose that player II plays $Y = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}).$ Find the optimal response strategy for player I.

Example 1.17 (cont'd)

- Solution
  - Let $X = (x_1, x_2, 1 - x_1 - x_2)$. Calculate
    $$E(X,Y) = XAY^T = -\frac{x_1}{4} - \frac{x_2}{2} + \frac{5}{4}.$$
  - $E(X,Y)$ is maximized by taking $x_1 = x_2 = 0$ and then necessarily $x_3 = 1$.
  - The best response strategy for player I if player II uses $Y = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ is $X^* = (0,0,1)$.
  - $E(X^*,Y) = \frac{5}{4}$, which is larger than the value of the game $v(A) = 1$.

Example 1.17 (cont'd)

- How player 1 should play if player II decides to deviate from the optimal Y.
- ***This shows that any deviation from a saddle could result in a better payoff for the opposing player.***
- If one player knows that the other player will not use her part of the saddle, then the best response may not be the strategy used in the saddle.
- In other words, if (X^*, Y^*) is a saddle point, the best response to $Y \neq Y^*$ may not be X^*, but some other X, even though it will be the case that $E(X^*, Y) \geq E(X^*, Y^*)$.

Analysis

- Because $E(X, Y)$ is linear in each strategy when the other strategy is fixed, the best response strategy for player I will usually be a <u>pure strategy</u>.
  - For instance, if *Y* is given, then $E(X, Y) = ax_1 + bx_2 + cx_3,$ for some values *a*, *b, c* that will depend on *Y* and the matrix.
  - The maximum payoff is then achieved by looking at the largest of *a*, *b*, *c*, and taking $x_i = 1$ for the x multiplying the largest of *a*, *b*, *c*, and the remaining values of $x_j = 0.$ In general,

$$\max\{ax_1 + bx_2 + cx_3 \mid x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 \geq 0\} \qquad (1.6.1)$$
$$= \max\{a, b, c\}$$

Analysis (cont'd)

– Suppose that $\max\{a,b,c\} = c$. Take $x_1 = 0, x_2 = 0, x_3 = 1,$ we get

$$\max\{ax_1 + bx_2 + cx_3 \mid x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 \geq 0\}$$
$$\geq a \cdot 0 + b \cdot 0 + c \cdot 1 = c.$$

– On the other hand, since $x_1 + x_2 + x_3 = 1,$ we see that

$$ax_1 + bx_2 + c(1 - x_1 - x_2) = x_1(a - c) + x_2(b - c) + c \leq c,$$

Since $a - c < 0, b - c < 0$ and $x_1, x_2 \geq 0.$

– We conclude that

$$c \geq \max\{ax_1 + bx_2 + cx_3 \mid x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 \geq 0\} \geq c,$$

and this establishes (1.6.1). This shows that $X^* = (0, 0, 1)$ is a best response to *Y*.

Analysis (cont'd)

- **It is possible to get a mixed strategy best response but only if <u>some or all of the coefficients *a, b, c* are equal</u>.**
  - For instance, if *b* = *c*, then

$$\max\{ax_1 + bx_2 + cx_3 \mid x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 \geq 0\} = \max\{a, c\}$$

  - Suppose that max{*a*, *c*} = *c*. We compute

$$\begin{aligned}
&\max\{ax_1 + bx_2 + cx_3 \mid x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 \geq 0\} \\
&= \max\{ax_1 + c(x_2 + x_3) \mid x_1 + x_2 + x_3 = 1\} \\
&= \max\{ax_1 + c(1 - x_1) \mid 0 \leq x_1 \leq 1\} \\
&= \max\{x_1(a - c) + c \mid 0 \leq x_1 \leq 1\} \\
&= c.
\end{aligned}$$

  - This maximum is achieved at $X^* = (0, x_2, x_3)$ for any $x_2 + x_3 = 1,$ $x_2 \geq 0, x_3 \geq 0,$ and we can get a mixed strategy as a best response.

Analysis (cont'd)

- In general, **if one of the strategies, say, *Y* is given and known,** then

$$\max_{X \in S_n} \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{m} a_{ij} y_j \right) = \max_{1 \le i \le n} \left(\sum_{j=1}^{m} a_{ij} y_j \right).$$

  In other words,

$$\max_{X \in S_n} E(X, Y) = \max_{1 \le i \le n} E(i, Y).$$

  We proved this in the proof of Theorem 1.3.7, part (e).

  – *Best response strategies are frequently used when we assume that the opposing player is Nature or some nebulous player that we think may be trying to oppose us like the market in an investment game.*

Example 1.18

- Suppose that player I has some money to invest with three options: stock(S), bonds(B), or CDs (certificates of deposit). The market (player II) can be in one of three states: good(G), neutral(N), or bad(B). Here is a possible game matrix in which the numbers represent the annual rate of return to the investor :

I/II	G	N	B
S	12	8	-5
B	4	4	6
CD	5	5	5

.

- Note that this game does not have a saddle in pure strategies.

Example 1.18 (cont'd)

- Assumption:
  - The market is the opponent with the goal of minimizing the investor's rate of return ➔ can be viewed as a two-personal zero sum game.
  - The market may be in any one of the three states with equal likelihood, then the market will play the strategy Y=(1/3, 1/3, 1/3).
  - Note that

- The response of player I
  - The investor seeks an *X** for which $E(X^*, Y) = \max_{X \in S_3} E(X, Y)$.
  - If we assume that the market is an opponent in a game then the value of the game is $v(A) = 5$.

Example 1.18 (cont'd)

- One of the optimal strategies: $X^* = (0, 0, 1), Y^* = (0, \frac{1}{2}, \frac{1}{2})$.
- If instead $Y = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, then the best response for player I is $X = (0, 0, 1)$, with payoff to I equal to 5.
- If $Y = (\frac{2}{3}, 0, \frac{1}{3})$, the best response is $X = (1, 0, 0)$, with payoff to I equal to $\frac{19}{3} > 5$.

Example 1.18 (cont'd)

- It may seem odd that the best response strategy in a zero sum two person game is usually a pure strategy.

- Suppose that someone is flipping a coin that is not fair—say heads comes up 75% of the time.
  - If you think it is 75% of the time, then you will be correct 75 x 75 = 56.25% of the time!
  - If you say heads **all the time,** you will be correct 75% of the time, and that is the best you can do.

Example 1.19

- Here is the matrix, assuming $\alpha, \beta, \gamma > 0$:

You/God	God exists	God doesn't exist
Believe	α	$-\beta$
Don't believe	$-\gamma$	0

.

  - $v^+ = 0$ and $v^- = \max(-\beta, -\gamma) < 0$, so this game does not have a saddle point in pure strategies unless $\beta = 0$ or $\gamma = 0$.
  - If you believe and God exist, then you receive the amount α from God.
  - If there is no loss or gain to you if you play don't believe, then that is what you should do, and God should play **not exist.** In this case the value of the game is zero.

Example 1.19 (cont'd)

– Let $Y = (y, 1-y)$ be an optimal strategy for God. Then it must be true that

$$E(1,Y) = \alpha y - \beta(1-y) = v(A) = -\gamma y = E(2,Y).$$

Solve and get the optimal strategy for God is

$$Y = \left(\frac{\beta}{\alpha+\beta+\gamma}, \frac{\alpha+\gamma}{\alpha+\beta+\gamma}\right)$$

and the value of the game to you is

$$v(A) = \frac{-\gamma\beta}{\alpha+\beta+\gamma} < 0.$$

Example 1.19 (cont'd)

– Your optimal strategy $X = (x, 1-x)$ must satisfy

$$E(X,1) = \alpha x - \gamma(1-x) = -\beta x = E(X,2) \implies$$

$$x = \frac{\gamma}{\alpha+\beta+\gamma}, \text{ and } X = \left(\frac{\gamma}{\alpha+\beta+\gamma}, \frac{\alpha+\beta}{\alpha+\beta+\gamma}\right).$$

– If γ, the penalty to you if you don't believe and God exists is loss of eternal life, represented by a very large number. In this case, the percent of time you play believe, $x = \gamma/(\alpha+\beta+\gamma)$ should be fairly close to 1, so you should play believe with high probability.

– If this is a zero sum game, God would then play doesn't exist with high probability.
  - It may not make much sense to think of this as a zero sum game.
  - Maybe we should just look at this like a best response for you, rather than as a zero sum game.

Example 1.19 (cont'd)

- Suppose that God plays the strategy $Y^0 = (\frac{1}{2}, \frac{1}{2})$. Find your best response strategy.
  - Calculate $f(x) = E(X, Y^0)$, where $X = (x, 1-x), 0 \le x \le 1$. We get

$$f(x) = x\,\frac{\alpha + \gamma - \beta}{2} - \frac{\gamma}{2}.$$

  - The maximum of $f(x)$ over $x \in [0,1]$ is

$$f(x^*) = \begin{cases} \frac{\alpha - \beta}{2}, & \text{at } x^* = 1 \text{ if } \alpha + \gamma > \beta; \\ -\frac{\gamma}{2}, & \text{at } x^* = 0 \text{ if } \alpha + \gamma < \beta; \\ -\frac{\gamma}{2}, & \text{at any } 0 \le x \le 1 \text{ if } \alpha + \gamma = \beta. \end{cases}$$

  - For $\gamma >> \beta$, the best response strategy would be $X^* = (1, 0)$.